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Optimum processing of continuous noisy measurement data

in a discrete Kalman filter

by

Delwyn Donald Bluhm

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I. INTRODUCTION

In communication and control work a large class of theo-
retical and practical problems deal with the separation of
random signals from random noise. These problems are solved
by applying linear estimation techniques where an optimal
estimate of a random signal, random variable, or control
parameter is determined. The optimal estimate is generated
from measurement data corrupted by additive noise.

Gauss (1) performed the first studies to determine least-
squares estimates of unknown parameters in the early nine-
teenth century. The next significant work dealing with esti-
mation of random signals was accomplished by Wiener (2) in the
1940's. This work showed that the time-domain approach to the
solution of particular linear estimation problems leads to the
integral equation called the Wiener-Hopf equation. The solu-
tion of this equation yields the optimum filter (Wiener
filter) to the so-called Wiener filter problem. When noisy
measurement data is applied to the optimum filter, the output
is an unbiased minimum variance estimate of the random signal.
The practical usefulness of the Wiener-Hopf equation for solu-
tions to the Wiener problem is limited for a number of
reasons: (i) The filter cannot be easily synthesized from
its impulse response specification which is the normal form
of the solution. (ii) Computer solution of the Wiener-Hopf
equation is generally not recommended for complex problems.
Bode & Shannon (3) attacked the Wiener filter problem in 1950

by a frequency-domain viewpoint. Considerable work in this area continued during the 1950's but the preceding limitations were not eliminated.

With the advent of the digital computer, interest in recursive least-square estimates was stimulated using differential or difference equations. Kalman (4) in 1960 introduced a new approach to the problem of linear filtering for random sequences (discrete case). Using the state-transition method he found that a single derivation applied to a very large class of problems. In 1961 Kalman and Bucy (5) extended the original method to random processes (continuous case) by deriving a matrix differential equation called the covariance equation whose solution completely specified the optimal filter. Thus the matrix differential equation was the transformed equivalent of the Wiener-Hopf integral equation. The former, however, could be readily solved on a digital computer. The new "Kalman filtering" approach to linear filtering eliminated the limitations encountered when using the Wiener-Hopf equation and has today proved its practical usefulness in aerospace and military systems.

The measurement of a random signal in the presence of additive noise can be performed continuously or at discrete intervals. In a Kalman filter, discrete or sampled measurements are linear functions of the "state" of the estimation problem corrupted by noise, and they are used to determine an optimal estimate of the state at the time of the measurement.

Continuous noisy measurement data processed in a Kalman-Bucy filter yields a minimum optimal estimation error which is approximately equivalent to the optimal estimation error resulting from a discrete Kalman filter with an infinitesimal sampling interval. For many applications it is desirable and more efficient from a computer standpoint to use the discrete Kalman filter with sampled measurements rather than the Kalman-Bucy filter even though continuous measurement data is available.

The possibility exists, when additive measurement noise is present, that the estimation error of the discrete random state can be reduced in a discrete Kalman filter if all the continuous measurement data is used to form a better discrete "sample" of the continuous data in lieu of simply accepting a raw sample. The object of this work is to explore this approach in discrete Kalman filtering.

Two specific methods are introduced for processing the continuous measurement data: interval-averaging and linearized-sampling. These processes yield discrete "samples" which when incorporated into the usual discrete Kalman filter produce a modified set of Kalman filter equations with delayed states as observables. The new Kalman filtering equations are used to analyze two examples which verify that the discrete estimation error can be reduced by preprocessing continuous measurement data.

II. REVIEW OF LITERATURE ON KALMAN FILTERING

Before reviewing the current literature on Kalman filtering, it might be well to define the basic problem of unbiased, minimum variance, linear estimation as given by Sorenson and Stubberud (6). The definition must be referred to a mathematical model as given by a linear dynamical system described by a linear, stochastic, vector differential equation of state evolution and by a measurement model supplying the only information about the state. Now, given all measurement data $\underline{z}(\tau)$ up to τ as a linear function of the state $\underline{x}(t)$, then define the unbiased, minimum variance linear (optimum) estimate as $\underline{\hat{x}}(t \mid \tau)$ where:

(i) $E[\underline{\hat{x}}(t \mid \tau)] = E[\underline{x}(t)]$.

This equation implies that $\overset{\wedge}{\underline{x}}(t \mid \tau)$ is unbiased.

(ii) Loss Function $= L \overset{\Delta}{=} E\{[\underline{x}(t)-\underline{\hat{x}}(t \mid \tau)]^T[\underline{x}(t)-\underline{\hat{x}}(t \mid \tau)]\}$

$= $ minimum.

The estimate $\underline{\hat{x}}(t \mid \tau)$ is optimum in the sense that the expected value of the square of the error magnitude is minimized when $\underline{\hat{x}}(t \mid \tau)$ is chosen to satisfy the loss equation.

The types of estimation problems are divided into three categories each based on the amount of measurement data $\underline{z}(\tau)$ available as described by τ.

Prediction problem: The optimal estimate of the state $\underline{x}(t)$ at some future time t is to be determined from all

data $\underline{z}(\tau)$ where $\tau<t$.

Filtering problem: The optimal estimate of the state $\underline{x}(t)$ at the present time t is to be determined from all data $\underline{z}(\tau)$ where $\tau=t$.

Smoothing problem: The optimal estimate of the state $\underline{x}(t)$ at some previous time t is to be determined from all data $\underline{z}(\tau)$ where $\tau>t$.

Following the publications by Kalman (4) and Kalman and Bucy (5) of their pioneering works on linear filtering and prediction problems many researchers entered the field. The applicability of their approach to computer solution of practical problems made Kalman filtering very popular. Many valuable contributions have been made that either clarify the basic work or broaden its applicability by the use of generalizations and extensions. Only the most significant contributions that directly affect this work will be mentioned.

The work by Lee (7) in 1964 derived the discrete Kalman filter in a far more straightforward way by eliminating the method of orthogonal projections which Kalman used in his discussion. He also presented the continuous Kalman filter model and solution in a very concise form. Additional insight and greater clarity of the discrete Kalman filtering process was provided by an unpublished work by Brown (8) in 1964 using a different approach to derive the Kalman equations.

The subject of Kalman filtering including theory, compu-

tational considerations, and applications was thoroughly
covered by Sorenson (9) in 1966. The major portion of the
presentation concerned the time-discrete model since the
author felt it is the most natural version for implementation
on a digital computer. His derivation of the Kalman filter
using state vector and state space notions was accomplished
in a manner which relied upon physical intuition. This pro-
vided much insight into linear estimation theory as developed
by Kalman and made the presentation more readable than some
earlier works. A simplified derivation for an unforced
dynamical system was developed first with extensions to deter-
ministic and random forcing functions, and correlated se-
quences. An interesting development of the Kalman-Bucy filter
equations for continuous dynamical systems and measurement
processes was introduced. By causing the sampling interval
to become infinitesimal in the discrete model, the resulting
continuous filter model involved differential rather than
difference equations and white noise processes rather than
white noise sequences.

Horton (10) in 1967 investigated one method of presmooth-
ing or averaging continuous measurements within discrete time
intervals before incorporating them into a discrete Kalman
filter. The derivation was limited to smoothing the continu-
ous a priori measurement error rather than only the measure-
ment which caused some difficulties with this technique.
Continuous measurement noise was realistic in that it was

assumed to be colored or Markov in character. A small range of permissible discrete time intervals was found to exist where this method of presmoothing yielded improved results over normal discrete sampling methods.

In the Kalman-Bucy (5) filter for continuous linear dynamic systems it was assumed that all measurement noise processes were gaussian and "white," i.e., noise with correlation times short compared to times of interest in the system. Clearly this is a restriction not always satisfied in practice, therefore it was deemed necessary to generalize their results for cases where measurement noise exhibits correlation between different instants of time i.e., the noise is "colored". Bryson and Johansen (11) in 1965 accomplished this generalization by introducing a "shaping filter" which simulated the colored noise from white-noise inputs. The colored-noise vector became a part of an augmented state variable vector and the measurements contained only linear combinations of the augmented state variables. This procedure reduced the more general problem of colored measurement noise to a problem of the type considered by Kalman and Bucy. This technique was clearly illustrated in several simple examples by Nahi (12). An optimal filtering problem with Gauss-Markov measurement noise was reduced to a problem of the Kalman and Bucy type by Stear and Stubberud (13) in 1968 without using a "shaping filter" and state vector augmentation.

Of course in the successful application of Kalman filter theory a paramount requirement is that the model must truly represent the physical situation. There are cases where the physical problem at hand does not fit the assumed format of discrete Kalman filter even after the generalization to "colored" measurement noise has been made. More specifically, consider the discrete estimation problem given observations which are functions of the integrals of the system states over a sequence of finite intervals rather than simply functions of the system states directly. This problem, which does not fit the prescribed format of the discrete Kalman filter, was considered by Brown and Hartmann (14) in 1968. A new relationship was presented which showed that the measurement was linearly related to the previous as well as the present states. The necessary recursive filter techniques were adapted to this situation.

In 1970 Sorenson and Stubberud (6) discussed the fundamental aspects of the unbiased, minimum variance, linear estimation problem, i.e., the theory of Kalman filtering, in depth. The presentation included complete derivations of the Kalman-Bucy filter equations, Kalman discrete filter equations, treatment of the colored measurement noise problem, and behavior aspects of the estimate.

III. THEORY OF KALMAN FILTERING

A. Dynamic System and Measurement Model

As described in the review of literature, the statement of the estimation problem must be referred to a mathematical model. Consider the linear, vector differential equation which describes the state of a continuous dynamical system

$$\dot{\underline{x}}(t) = F(t) \, \underline{x}(t) + \underline{w}(t) \qquad (3.1)$$

where

$\underline{x}(t)$ is the n-vector of state variables or state vector

$F(t)$ is an nxn plant matrix with time continuous elements

$\underline{w}(t)$ is an n-dimensional, gaussian white-noise process or plant driving function.

Let $\underline{w}(t)$ have the statistics

$$E\{\underline{w}(t)\} = 0 \qquad \text{for all t}$$
$$E\{\underline{w}(t_1)\underline{w}^T(t_2)\} = Q(t)\delta(t_1-t_2)$$

where $Q(t)$ is an nxn symmetric matrix and $\delta(t_1-t_2)$ is the Dirac delta function or impulse function.

The relationship between the state vector $\underline{x}(t)$ and the only available information about the state defined $\underline{z}(t)$ for m-vector of measurements is given by the measurement model equation as

$$\underline{z}(t) = M(t) \, \underline{x}(t) + \underline{v}(t) \qquad (3.2)$$

where

$M(t)$ is an mxn observation matrix with time continuous elements

$\underline{v}(t)$ is the measurement noise and is an m-dimensional gaussian white-noise process.

Let $\underline{v}(t)$ have the statistics

$$E\{\underline{v}(t)\} = 0 \qquad \text{for all } t$$

$$E\{\underline{v}(t_1)\underline{v}^T(t_2)\} = R(t)\delta(t_1-t_2)$$

where $R(t)$ is an mxm symmetric matrix and $\delta(t_1-t_2)$ is the Dirac delta function. Generally the plant noise $\underline{w}(t)$ and the measurement noise $\underline{v}(t)$ are considered to be independent.

The general solution of Equation 3.1 is

$$\underline{x}(t) = \phi(t,t_0)\ \underline{x}(t_0) + \int_{t_0}^{t} \phi(t,\tau)\underline{w}(\tau)d\tau \qquad (3.3)$$

where $\phi(t,\tau)$, the state transition matrix, is the solution of the matrix differential equation

$$\frac{d\phi(t,\tau)}{dt} = F(t)\ \phi(t,\tau) \qquad \text{for all } \tau \qquad (3.4)$$

Of course Equations 3.2 and 3.3 can be combined yielding the general form of the measurement model. There are several important properties of the state transition matrix which will be used later in this investigation.

Property: 1

$$\phi(\tau,\tau) \overset{\Delta}{=} I \qquad \text{for all } \tau \qquad (3.5)$$

Property: 2

$$\phi(t_2,t_0) = \phi(t_2,t_1)\phi(t_1,t_0) \qquad (3.6)$$

Property: 3

$$\phi(t_1,t_2) = \phi^{-1}(t_2,t_1) \qquad (3.7)$$

For the linear fixed system, where the plant matrix F is a
constant nxn coefficient matrix, the calculation of the state
transition matrix $\phi(t)$ may be performed by the frequency-
domain method where

$$\phi(t) = \mathcal{L}^{-1}[sI-F]^{-1} \qquad (3.8)$$

This method is generally the most convenient for fixed systems
even though the inverse of [sI-F] may be difficult to deter-
mine.

The physical situation may now be mathematically modeled
by the dynamical system and the measurement model Equations
3.1 and 3.2 respectively. The state transition matrix de-
scribes the transition of the state of the system in that it
describes the motion of the state vector in state space from
its initial position at t_0 to its final position at t. The
first term of the general solution Equation 3.3 represents
the initial condition response of the system state variables
by projecting the state at t_0 through the transition matrix
while the second term represents the forced response due to
the white noise driving functions. The latter response term
creates an uncertainty in the actual value of the state vector
at time t. Perfect measurements of each state variable at
time t could cancel this uncertainty. However, physical
measurements with infinite precision can never be made; in
addition direct physical measurement of some state variables
is often not possible. A filter is therefore required to

determine the "best" estimate in some sense of the state vector from available measurement data and thus reduce but not necessarily eliminate the uncertainty about the true value of the state vector at time t. The Kalman filter is a technique devised to solve the linear estimation problem in this manner.

B. Time-Discrete Kalman Filtering

The discrete Kalman filter is associated with a mathematical model; however, in this case linear vector difference equations are specified. The state vector of a dynamical system at time t_k is given by the equation

$$\underline{x}(t_k) = \phi(t_k, t_{k-1})\underline{x}(t_{k-1}) + \underline{w}_{k-1}$$

or using simplified notation

$$\underline{x}_k = \phi_{k-1} \, \underline{x}_{k-1} + \underline{w}_{k-1} \tag{3.9}$$

where

$\underline{x}_k$ is the n-vector of state variables or state vector at time t_k

ϕ_{k-1} is the state transition matrix over the interval (t_k, t_{k-1})

$\underline{w}_{k-1}$ is the plant noise and is an n-dimensional vector random sequence.

From Equations 3.1, 3.3 and 3.9 it can be shown that

$$\underline{w}_{k-1} = \int_{t_{k-1}}^{t_k} \phi(t_k, \tau)\underline{w}(\tau)d\tau \tag{3.10}$$

Measurement data $\underline{z}_k$ are obtained at discrete instants of time t_k and this information is assumed to be related to the state vector by the measurement model equation

$$\underline{z}_k = M_k \underline{x}_k + \underline{v}_k \qquad (3.11)$$

where

M_k is an mxn observation matrix

$\underline{v}_k$ is the measurement noise and is an m-dimensional vector random sequence

$\underline{v}_k$ and $\underline{w}_k$ are assumed to be uncorrelated.

Given the model Equations 3.9 and 3.11 the recursive Kalman filter must yield an estimate $\hat{\underline{x}}_k$ of the state vector at t_k that is a linear combination of an estimate at t_{k-1} and the measurement data $\underline{z}_k$. This estimate must be optimum in the sense that

$$E\{(\hat{\underline{x}}_k - \underline{x}_k)^T (\hat{\underline{x}}_k - \underline{x}_k)\} = \text{minimum value} \qquad (3.12)$$

Stating the above mathematically by defining an unknown gain matrix, which will be chosen later to optimize the estimate, yields the Kalman discrete filter equation

$$\hat{\underline{x}}_k = \hat{\underline{x}}_k' + K_k (\underline{z}_k - \hat{\underline{z}}_k') \qquad (3.13)$$

where

$\hat{\underline{x}}_k$ is the a posteriori estimate of the state vector at time t_k

$\hat{\underline{x}}_k'$ is the a priori estimate of the state vector at time t_k

K_k is the optimal gain matrix at time t_k

$\underline{z}_k$ is the measurement data at time t_k

$\underline{\hat{z}}'_k$ is the a priori estimate of the measurement value at time t_k.

The first term of Equation 3.13 is the predicted estimate of the state vector at t_k since

$$\underline{\hat{x}}'_k = \phi_{k-1} \, \underline{\hat{x}}_{k-1} \tag{3.14}$$

when no measurement information is available. The expected measurement value or a priori estimate $\underline{\hat{z}}'_k$ is

$$\underline{\hat{z}}'_k = M_k \, \underline{\hat{x}}'_k = M_k \, \phi_{k-1} \, \underline{\hat{x}}_{k-1} \tag{3.15}$$

This is combined with the measurement data in the second term of Equation 3.13 to modify and correct the original estimate given by the first term.

The derivation of the Kalman filtering equations will not be completely presented here. Only definitions, key steps and any physical insight necessary to total understanding of the final results will be given. Numerous references treat this subject in depth. For example, see Kalman (4), Lee (7), and Sorenson (9). Several definitions are required.

$\underline{e}_k \overset{\Delta}{=} \underline{\hat{x}}_k - \underline{x}_k$ = a posteriori estimation error in the state vector

$\underline{e}'_k \overset{\Delta}{=} \underline{\hat{x}}'_k - \underline{x}_k$ = a priori estimation error in the state vector

$P_k \overset{\Delta}{=} E\{\underline{e}_k \, \underline{e}_k^T\}$ = covariance matrix of a posteriori estimation error

$$P_k^* \overset{\Delta}{=} E\{\underline{e}_k' \ \underline{e}_k'^T\} = \text{covariance matrix of a priori estima-}$$

$$\text{tion error}$$

$$L \overset{\Delta}{=} \text{trace } P_k = E\{\underline{e}_k^T \underline{e}_k\} = \text{loss function}$$

$$V_k \overset{\Delta}{=} E\{\underline{v}_k \ \underline{v}_k^T\} = \text{covariance matrix of measurement noise}$$

$$\text{sequence}$$

$$H_{k-1} \overset{\Delta}{=} E\{\underline{w}_{k-1} \ \underline{w}_{k-1}^T\} = \text{covariance matrix of plant noise}$$

$$\text{sequences}$$

$$\Delta t \overset{\Delta}{=} t_k - t_{k-1} = \text{time interval}$$

Using the Kalman discrete filter Equation 3.13 and the above definitions allows $\underline{e}_k$ to be formed as

$$\underline{e}_k = \hat{\underline{x}}_k - \underline{x}_k = (I - K_k M_k) \ \underline{e}_k' + K_k \ \underline{v}_k \qquad (3.16)$$

Noting that Equation 3.12 for the optimal estimate can be re-written as the minimum value of the loss function implies that the loss function should be formed using Equation 3.16. The optimum gain matrix is then determined by minimizing the loss function with respect to K_k by letting

$$\frac{\partial L}{\partial K_k} = 0 \qquad (3.17)$$

The result is

$$K_k = P_k^* M_k^T (M_k \ P_k^* \ M_k^T + V_k)^{-1} \qquad (3.18)$$

Again using Equation 3.16 to form the matrix $E\{\underline{e}_k \ \underline{e}_k^T\}$ after noting that $\underline{e}_k'$ and $\underline{v}_k$ are uncorrelated reduces to

$$P_k = P_k^* - K_k(M_k \ P_k^* \ M_k^T + V_k)K_k^T \qquad (3.19)$$

Forming the matrix $E\{\underline{e}_k' \ \underline{e}_k'^T\}$, using the fact that

$$\underline{e}_k' = \hat{\underline{x}}_k' - \underline{x}_k = \phi_{k-1}\, \underline{e}_{k-1} - \underline{w}_{k-1} \qquad (3.20)$$

from Equations 3.9 and 3.14, produces the equation

$$P_k^* = \phi_{k-1}\, P_{k-1}\, \phi_{k-1}^T + H_{k-1} \qquad (3.21)$$

Thus the discrete Kalman filter is defined by Equations 3.13, 3.14, 3.18, 3.19 and 3.21.

C. Kalman Filtering with Delayed States as Observables

Recursive filtering techniques can be applied to a random process even when the observable has a linear relationship to the previous as well as the present state variables (14). Typically the physical situation may be faithfully represented by the dynamical model Equation 3.9 but does not conform to the measurement model Equation 3.11 as assumed in the discrete Kalman filter. Consider a measurement process where only the integrals of state variables over a sequence of finite time intervals are available. Now define a new state which is equal to the integral of the former state. Then

$$\text{measurement} = \int_{t_{k-1}}^{t_k} (\text{former state}) + \text{noise}$$

and

$$\text{measurement} = \text{new state}\Big|_{t=t_k} - \text{new state}\Big|_{t=t_{k-1}}$$
$$+ \text{noise} \qquad (3.22)$$

Equation 3.22 can be generalized to form a revised measurement model.

$$\underline{z}_k = M_k\, \underline{x}_k + N_k\, \underline{x}_{k-1} + \underline{v}_k \qquad (3.23)$$

which shows the measurement data at t_k is linearly dependent
on the present state at t_k and on the previous state at t_{k-1}.

The new mathematical model, Equations 3.9 and 3.23, can
be transformed into the format of the original Kalman filter
model, Equations 3.9 and 3.11, by employing a double-state
approach. The state vectors $\underline{x}_k$ and $\underline{x}_{k-1}$ are combined into a
new state vector and the usual Kalman filter equations apply.
A more direct or straightforward approach is to derive a new
set of recursive equations for the new mathematical model
just as was done originally for the Kalman filter.

The major deviation in this derivation is in the inter-
pretation of the a priori estimate $\hat{\underline{z}}_k'$ which will always be
the optimal estimate of $\underline{z}_k$ based on all measurement data up
through $\underline{z}_{k-1}$. Therefore

$$\hat{\underline{z}}_k' = M_k \hat{\underline{x}}_k' + N_k \hat{\underline{x}}_{k-1} = M_k \phi_{k-1} \hat{\underline{x}}_{k-1} + N_k \hat{\underline{x}}_{k-1} \qquad (3.24)$$

where all definitions made previously still apply and where

$$\underline{x}_k = \phi_{k-1} \underline{x}_{k-1} + \underline{w}_{k-1} \qquad (3.9)$$

$$\underline{z}_k = M_k \underline{x}_k + N_k \underline{x}_{k-1} + \underline{v}_k \qquad (3.23)$$

$$\underline{e}_k = \hat{\underline{x}}_k - \underline{x}_k \qquad (3.16)$$

$$\underline{e}_k' = \hat{\underline{x}}_k' - \underline{x}_k = \phi_{k-1} \underline{e}_{k-1} - \underline{w}_{k-1} \qquad (3.20)$$

$$\hat{\underline{x}}_k = \hat{\underline{x}}_k' + b_k(\underline{z}_k - \hat{\underline{z}}_k') \qquad (3.25)$$

Observe that Equation 3.25 is identical to Equation 3.13
except the optimum gain matrix is now denoted b_k. As before

the quantities $\underline{e}_k$, P_k, and L are formed so that the loss function can be minimized with respect to the optimum gain matrix b_k. Since the vector $\underline{e}_k'$ is unchanged P_k^* remains the same as given before by Equation 3.21.

Thus the recursive Kalman filtering equations with delayed states as observables are found to be

$$\hat{\underline{x}}_k = \hat{\underline{x}}_k' + b_k(\underline{z}_k - \hat{\underline{z}}_k') \tag{3.25}$$

$$\hat{\underline{x}}_k' = \phi_{k-1} \hat{\underline{x}}_{k-1} \tag{3.14}$$

$$P_k^* = \phi_{k-1} P_{k-1} \phi_{k-1}^T + H_{k-1} \tag{3.21}$$

$$Q_k = (M_k P_k^* M_k^T + V_k) + N_k P_{k-1} N_k^T + N_k P_{k-1} \phi_{k-1}^T M_k^T$$
$$+ M_k \phi_{k-1} P_{k-1} N_k^T \tag{3.26}$$

$$b_k = (P_k^* M_k^T + \phi_{k-1} P_{k-1} N_k^T) Q_k^{-1} \tag{3.27}$$

$$P_k = P_k^* - b_k Q_k b_k^T \tag{3.28}$$

IV. OPTIMUM DISCRETE PROCESSING OF CONTINUOUS MEASUREMENTS

The measurement of a random signal in the presence of additive noise can be performed continuously or at discrete intervals. A continuous minimum variance estimate $\hat{\underline{x}}(t \,|\tau)$ of the state vector $\underline{x}(t)$ is obtained from the Kalman-Bucy filter when continuous noisy measurement data is available. A discrete minimum variance estimate $\hat{\underline{x}}(t_k)$ is determined from the Kalman filter when the only available information about the states are discrete noisy measurements i.e., measurements obtained at discrete instants of time. Of course the discrete estimate may be almost as good as the continuous estimate if measurements are taken frequently. A natural extension of this is to convert noisy continuous measurement data into discrete form by sampling for processing in a discrete Kalman filter.

Assuming that the measurement data is a combination of both continuous and discrete physical processes it can be demonstrated that shortcomings exist when using either filter exclusively. Using all the data in a Kalman-Bucy filter will produce an estimate based solely on the continuous portion of the data since this filter cannot process discrete data. In this case all the discrete information is lost. Processing all the data by the discrete Kalman filter method using a sampling technique will produce estimates based on both continuous and discrete portions of the data. The estimation error

will depend considerably on the sampling frequency. It would
also be necessary to sample the continuous data coincidentally
with any discrete data available and to process the discrete
data as outlined in Chapter III. Even with these considera-
tions some of the available information in the continuous
portion of the data will be lost if the sampling interval is
not unrealistically small.

Considering the fact that the Kalman filter is composed
of a group of recursive equations which are particularly well
suited to implementation on the digital computer, all estima-
tion may be restricted to the discrete Kalman filtering method.
This being the case, it may be possible to form a better dis-
crete value of the continuous noisy measurement data in lieu
of simply sampling the data. This improved value would then
serve as the discrete measurement in the usual Kalman filter
equations.

In particular it is the measurement noise which prevents
the elimination of all uncertainty about the observable states.
Thus, if the effects of measurement noise in continuous data
could be reduced, some reduction in the state vector estima-
tion error could be expected. Smoothing the measurement data
is certainly one method of separating the true measurement
signal from the measurement noise. The general philosophy of
James, Nichols, and Phillips (15) can be applied here equally
as well as they used it for servomechanisms. In applying
their method the basic form of the smoothing process is

intuitively selected. The output of this process is incor-
porated in the discrete Kalman filter where the expected value
of the square of the error magnitude is minimized with respect
to the gain matrix. In one sense the smoothing process could
be thought of as a "prefiltering" process prior to use of the
Kalman filter.

It should be pointed out before proceeding that the re-
duced state vector estimation error will be optimum for the
particular smoothing process selected. It is not the absolute
minimum estimation error possible since the smoothing process
was chosen intuitively before the optimization process was
applied.

Several comments should be made regarding the smoothing
process. The smoothing must take place over the continuous
finite time interval. The output of the smoothing process
must be a discrete value to be of any practical value in a
discrete filter. Prefiltering the measurement data will
significantly change the usual Kalman filter model; therefore,
new equations must be derived. Hopefully the prefilter will
reduce the state vector estimation error without appreciably
increasing instrumentation and computer costs and will more
effectively use all the available continuous data.

Since in physical situations the measurement noise sta-
tistics may be white noise and even more realistically may be
colored, both cases will be considered in generalized form and

by example. The white noise case is included because in general the colored case is the more difficult problem and more costly for the computer to solve. Therefore, if correlation time is quite small, it may be sufficient to assume white measurement noise. This assumption is possible because white noise implies zero time correlation.

A. Interval-Averaging Data Containing White Noise

As a first choice select a simple integrating process which averages the continuous noisy measurement data over the time interval. The output of the filter is a discrete value as required. In addition, the process will smooth the continuous measurements to reduce unwanted measurement noise. Define the interval-averaging process as

$$\underline{z}_k \overset{\Delta}{=} \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \underline{z}(t) \ dt \tag{4.1}$$

where $\underline{z}(t)$ is the continuous measurement data. Observe that the prefilter is smoothing only the measurement and not the a priori estimation error. Using the a priori estimation error creates a basic problem which is discussed by Horton (10).

A block diagram for a generalized system is shown in Figure 1 for the case where the plant and measurement noise are both white noise. The continuous dynamical system and measurement model equations referring to this diagram are respectively
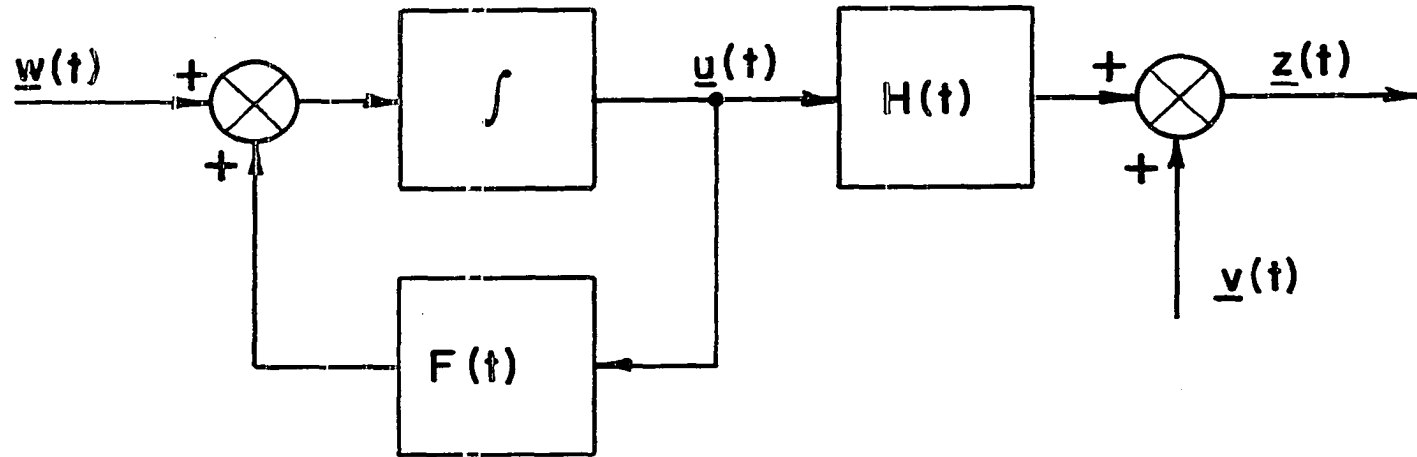
Figure 1.   Block diagram for generalized system with white measurement noise $\underline{v}(t)$

$$\underline{\dot{u}}(t) = F(t) \underline{u}(t) + \underline{w}(t) \tag{4.2}$$

$$\underline{z}(t) = H(t) \underline{u}(t) + \underline{v}(t) \tag{4.3}$$

where all quantities are defined as in Chapter III except that the state vector is now called $\underline{u}(t)$ and the observation matrix is $H(t)$.

The derivation of the new discrete Kalman filtering equations is begun by substitution of Equation 4.3 into Equation 4.1. The result is

$$\underline{z}_k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} [H(t) \underline{u}(t) + \underline{v}(t)] \, dt$$

$$= \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} H(t) \underline{u}(t) \, dt + \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \underline{v}(t) dt \tag{4.4}$$

The latter term defined $\delta \underline{z}_k$ will remain the modified discrete noise contribution since in this case the noise $\underline{v}(t)$ is not being treated as a state variable. The averaging method also eliminates the problem of infinite variance which results from sampling data containing white noise. The former term is treated by defining a new state

$$\underline{y}(t) \overset{\Delta}{=} \int H(t) \underline{u}(t) \, dt \tag{4.5}$$

so Equation 4.4 becomes

$$\underline{z}_k = \frac{1}{\Delta t} [\underline{y}(t_k) - \underline{y}(t_{k-1}] + \delta \underline{z}_k \tag{4.6}$$

where

$$\delta \underline{z}_k \overset{\Delta}{=} \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \underline{v}(t) \, dt \tag{4.7}$$

By differentiating

$$\underline{u}(t) = \underline{u}(t)$$

$$\underline{y}(t) = \int H(t)\ \underline{u}(t)\ dt$$

yields

$$\dot{\underline{u}}(t) = F(t)\ \underline{u}(t) + \underline{w}(t) \tag{4.2}$$

$$\dot{\underline{y}}(t) = H(t)\ \underline{u}(t) \tag{4.8}$$

and the new continuous plant model becomes

$$\begin{bmatrix} \dot{\underline{u}}(t) \\ \dot{\underline{y}}(t) \end{bmatrix} = \begin{bmatrix} F(t) & 0 \\ H(t) & 0 \end{bmatrix} \begin{bmatrix} \underline{u}(t) \\ \underline{y}(t) \end{bmatrix} + \begin{bmatrix} \underline{w}(t) \\ 0 \end{bmatrix} \tag{4.9}$$

The state transition matrix for Equation 4.9 is defined by

$$\frac{d\phi(t,t_{k-1})}{dt} \triangleq \begin{bmatrix} F(t) & 0 \\ H(t) & 0 \end{bmatrix} \phi(t,t_{k-1}) \tag{4.10}$$

and by

$$\phi_{k-1} \triangleq \phi(t,t_{k-1}) \Big|_{t=t_k} \tag{4.11}$$

Finally for the white measurement noise system given by Equations 4.2 and 4.3 the new generalized discrete plant model is

$$\begin{bmatrix} \underline{u}_k \\ \underline{y}_k \end{bmatrix} = \phi_{k-1} \begin{bmatrix} \underline{u}_{k-1} \\ \underline{y}_{k-1} \end{bmatrix} + \int_{t_{k-1}}^{t_k} \phi(t_k,\tau) \begin{bmatrix} \underline{w}(\tau) \\ 0 \end{bmatrix} d\tau \tag{4.12}$$

and from Equation 4.6 the generalized discrete measurement model is

$$\underline{z}_k = [0\ \ \tfrac{1}{\Delta t}] \begin{bmatrix} \underline{u}_k \\ \underline{y}_k \end{bmatrix} + [0\ -\tfrac{1}{\Delta t}] \begin{bmatrix} \underline{u}_{k-1} \\ \underline{y}_{k-1} \end{bmatrix} + \delta\underline{z}_k \tag{4.13}$$

where

$$\delta \underline{z}_k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \underline{v}(t) \, dt \qquad (4.14)$$

The generalized mathematical model Equations 4.12, 4.13 and 4.14 are in exactly the same form as the Kalman filter with delayed states given in Chapter III where

$$\underline{x}_k = \phi_{k-1} \, \underline{x}_{k-1} + \underline{w}_{k-1} \qquad (3.9)$$

$$\underline{z}_k = M_k \, \underline{x}_k + N_k \, \underline{x}_{k-1} + \underline{v}_k \qquad (3.23)$$

The new equations were generated directly from the generalized continuous system Equations 4.2 and 4.3 when averaging continuous noisy measurements containing white noise.

B. Interval-Averaging Data Containing Colored Noise

It has been shown that the colored-noise (i.e., noise which exhibits correlation at different instants of time) problem can be successfully approached when this noise can be described by a shaping filter driven by white noise. The problem is then reformulated by state vector augmentation to obtain a system in which only white noise appears explicitly. Thus the colored-noise problem will then fit the format of the discrete Kalman filter model. Treating the noise as a state variable yields a measurement model with no measurement noise term.

The block diagram in Figure 1 must be revised with the addition of a shaping filter as shown by Figure 2 for the colored measurement noise case. The continuous system equa-
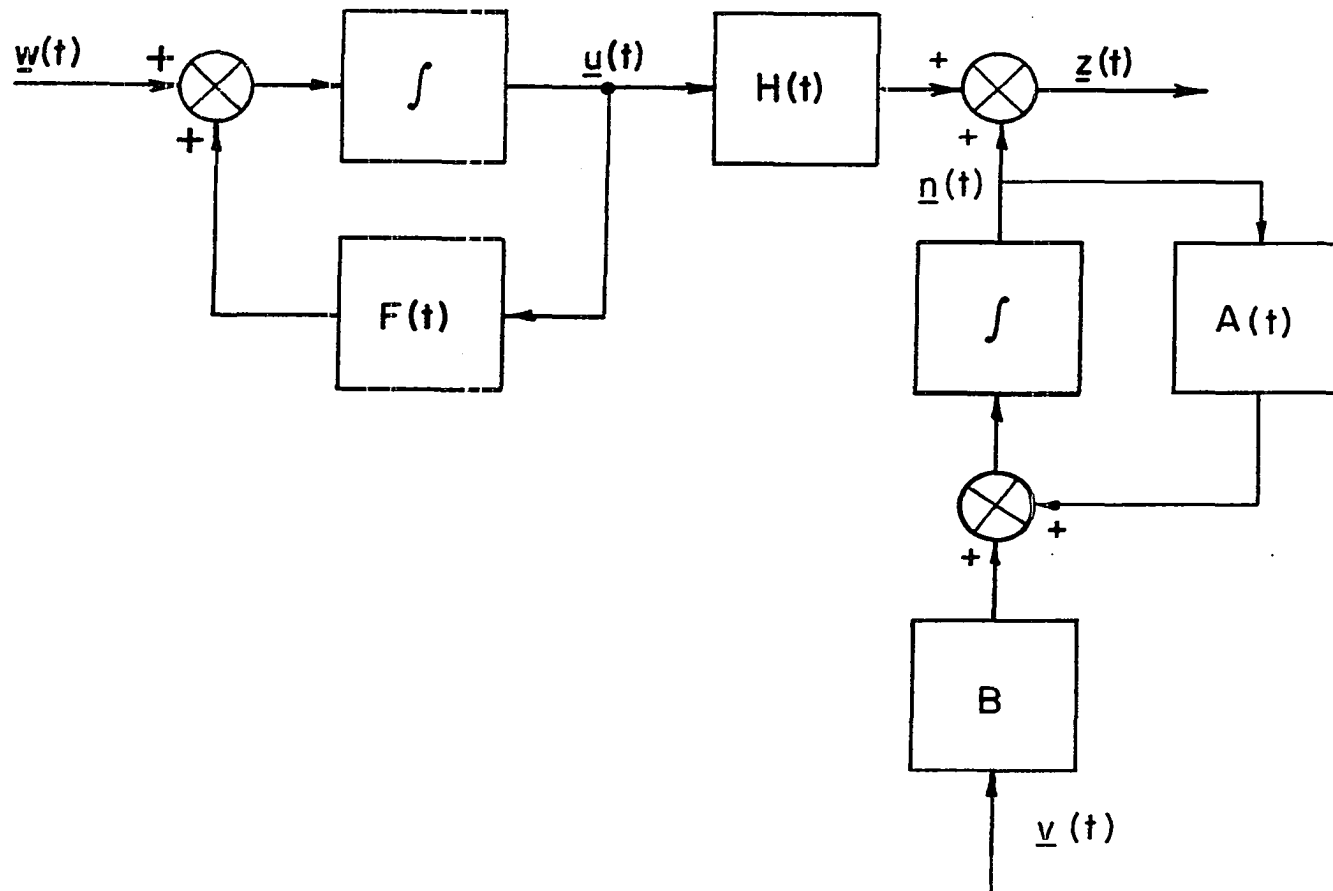
Figure 2. Block diagram for generalized system with colored measurement noise $\underline{n}(t)$ and shaping filter

tions corresponding to Figure 2 before state vector augmentation are

$$\dot{\underline{u}}(t) = F(t) \underline{u}(t) + \underline{w}(t) \tag{4.15}$$

$$\underline{z}(t) = H(t) \underline{u}(t) + \underline{n}(t) \tag{4.16}$$

The noise $\underline{n}(t)$ is a zero mean colored-noise process described by the shaping filter

$$\dot{\underline{n}}(t) = A(t) \underline{n}(t) + B\underline{v}(t) \tag{4.17}$$

where $A(t)$ and the statistics of the white noise $B\underline{v}(t)$ are chosen so that $\underline{n}(t)$ has the desired statistical character. State vector augmentation yields the reformulated system

$$\begin{bmatrix} \dot{\underline{u}}(t) \\ \dot{\underline{n}}(t) \end{bmatrix} = \begin{bmatrix} F(t) & 0 \\ 0 & A(t) \end{bmatrix} \begin{bmatrix} \underline{u}(t) \\ \underline{n}(t) \end{bmatrix} + \begin{bmatrix} & \underline{w}(t) \\ B & \underline{v}(t) \end{bmatrix} \tag{4.18}$$

$$\underline{z}(t) = [H(t) \quad I] \begin{bmatrix} \underline{u}(t) \\ \underline{n}(t) \end{bmatrix} + 0 \tag{4.19}$$

where it is assumed $\underline{v}(t)$ and $\underline{w}(t)$ are uncorrelated white-noise processes. Observe that the measurements in the augmented system are perfect since the measurement error term is zero.

The equivalent discrete measurement is determined by combining Equations 4.1 and 4.19 to obtain

$$\underline{z}_k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} H(t) \underline{u}(t) \, dt + \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \underline{n}(t) \, dt + \delta \underline{z}_k \tag{4.20}$$

where $\delta \underline{z}_k = 0$.

It might be well to point out here that the term

involving $\underline{n}(t)$ of Equation 4.20 must be simplified in terms of a new state variable just as will be done for the term preceding it since the measurement noise is being treated as a state variable creating perfect measurements. Thus this term is not a noise term in Equation 4.20 because the noise term is zero. Therefore define two new states

$$\underline{y}(t) \overset{\Delta}{=} \int H(t) \underline{u}(t) \, dt \qquad (4.21)$$

$$\underline{s}(t) \overset{\Delta}{=} \int \underline{n}(t) \, dt \qquad (4.22)$$

Substitution of Equations 4.21 and 4.22 into 4.20 simplifies to

$$\underline{z}_k = \frac{1}{\Delta t}[\underline{y}(t_k) - \underline{y}(t_{k-1})] + \frac{1}{\Delta t}[\underline{s}(t_k) - \underline{s}(t_{k-1})] + 0 \qquad (4.23)$$

By differentiating $\underline{u}(t)$, $\underline{n}(t)$, $\underline{y}(t)$ and $\underline{s}(t)$ yields

$$\underline{\dot{u}}(t) = F(t) \underline{u}(t) + \underline{w}(t) \qquad (4.15)$$

$$\underline{\dot{n}}(t) = A(t) \underline{n}(t) + B \underline{v}(t) \qquad (4.17)$$

$$\underline{\dot{y}}(t) = H(t) \underline{u}(t) \qquad (4.24)$$

$$\underline{\dot{s}}(t) = \underline{n}(t) \qquad (4.25)$$

which implies that the new continuous plant model is given by the equation

$$
\begin{bmatrix} \underline{\dot{u}}(t) \\ \underline{\dot{n}}(t) \\ \underline{\dot{y}}(t) \\ \underline{\dot{s}}(t) \end{bmatrix} = \begin{bmatrix} F(t) & 0 & 0 & 0 \\ 0 & A(t) & 0 & 0 \\ H(t) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}(t) \\ \underline{n}(t) \\ \underline{y}(t) \\ \underline{s}(t) \end{bmatrix} + \begin{bmatrix} \underline{w}(t) \\ B \quad \underline{v}(t) \\ 0 \\ 0 \end{bmatrix} \qquad (4.26)
$$

Define the state transition matrix as

$$\frac{d}{dt} \, \phi(t,t_{k-1}) \overset{\Delta}{=} \begin{bmatrix} F(t) & 0 & 0 & 0 \\ 0 & A(t) & 0 & 0 \\ H(t) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \phi(t,t_{k-1}) \qquad (4.27)$$

and ϕ_{k-1} as in Equation 4.11.

Thus using Equation 4.26 the new discrete plant model for the colored measurement noise case given by Equations 4.15, 4.16, and 4.17 is

$$\begin{bmatrix} \underline{u}_k \\ \underline{n}_k \\ \underline{y}_k \\ \underline{s}_k \end{bmatrix} = \phi_{k-1} \begin{bmatrix} \underline{u}_{k-1} \\ \underline{n}_{k-1} \\ \underline{y}_{k-1} \\ \underline{s}_{k-1} \end{bmatrix} + \int_{t_{k-1}}^{t_k} \phi(t_k,\tau) \begin{bmatrix} B & \underline{w}(\tau) \\ & \underline{v}(\tau) \\ & 0 \\ & 0 \end{bmatrix} d\tau \qquad (4.28)$$

and the new discrete measurement model is given by Equation 4.23 as

$$\underline{z}_k = [0 \ 0 \ \tfrac{1}{\Delta t} \ \tfrac{1}{\Delta t}] \begin{bmatrix} \underline{u}_k \\ \underline{n}_k \\ \underline{y}_k \\ \underline{s}_k \end{bmatrix} + [0 \ 0 \ \tfrac{-1}{\Delta t} \ \tfrac{-1}{\Delta t}] \begin{bmatrix} \underline{u}_{k-1} \\ \underline{n}_{k-1} \\ \underline{y}_{k-1} \\ \underline{s}_{k-1} \end{bmatrix} + 0 \qquad (4.29)$$

The generalized mathematical model, Equations 4.28 and 4.29, form a discrete Kalman filter with delayed states as

observables. It was derived directly from the generalized
continuous system described by Equations 4.15 and 4.16 when
averaging continuous measurements containing colored noise
defined by Equation 4.17.

C. Sampling Optimum-Linearized Data Containing White Noise

The integrating prefilter previously selected is defi-
nitely one of the more common techniques to average continuous
data; however, it does have several shortcomings. Its most
serious deficiency is that it does not allow for any drift or
change of the noise-free measurement value over the time
interval. In other words, the constant measurement value
resulting from the averaging prefilter could be thought of as
the "best" equivalent measurement value not only at t_k but at
any time t in the interval t_{k-1} to t_k. This obviously is not
the case if the noise-free measurement value does in fact
change over the time interval Δt. If the noise-free measure-
ment value were almost constant, the simple averaging pre-
filter would be excellent.

As a second choice consider a prefiltering process which
assumes that the noise-free measurement value does change
over Δt. In fact, assume that its change is approximately
linear over the time interval. Then the noisy measurement
data $\underline{z}(t)$ will reflect this trend over the time interval.
Now a linear least square approximation can be formed for
each of the p measurements in $\underline{z}(t)$ which of course is a p-

vector here. Thus let the ith continuous noisy measurement $z_i(t)$ be approximated by

$$L_i(m_i, t, b_i) = (t - t_{k-1})m_i + b_i \qquad (4.30)$$

This is shown in Figure 3 where constants m_i and b_i are chosen such that

$$\int_{t_{k-1}}^{t_k} [z_i(t) - L_i(m_i, t, b_i)]^2 \, dt = \text{minimum} \qquad (4.31)$$

which gives a least square approximation of L_i for $z_i(t)$. Combining Equations 4.30 and 4.31 defines the equation

$$F_i(m_i, b_i) \overset{\Delta}{=} \int_{t_{k-1}}^{t_k} [z_i(t) - (t - t_{k-1})m_i - b_i]^2 \, dt \qquad (4.32)$$

Now minimize F_i with respect to m_i and b_i. This yields

$$\frac{\partial F_i}{\partial m_i} = -2 \int_{t_{k-1}}^{t_k} [z_i(t) - m_i(t - t_{k-1}) - b_i][t - t_{k-1}] dt = 0 \qquad (4.33)$$

$$\frac{\partial F_i}{\partial b_i} = -2 \int_{t_{k-1}}^{t_k} [z_i(t) - m_i(t - t_{k-1}) - b_i] dt = 0 \qquad (4.34)$$

Equations 4.33 and 4.34 can now be solved simultaneously for m_i and b_i as functions of $z_i(t)$.

Note that Equation 4.30 for $L_i(m_i, b_i)$ could have been a vector equation $\underline{L}(\underline{m}, \underline{b})$ where the ith element is $L_i(m_i, b_i)$. Then the scalar quantity F, Equation 4.32, could have been defined as a scalar in terms of vectors as
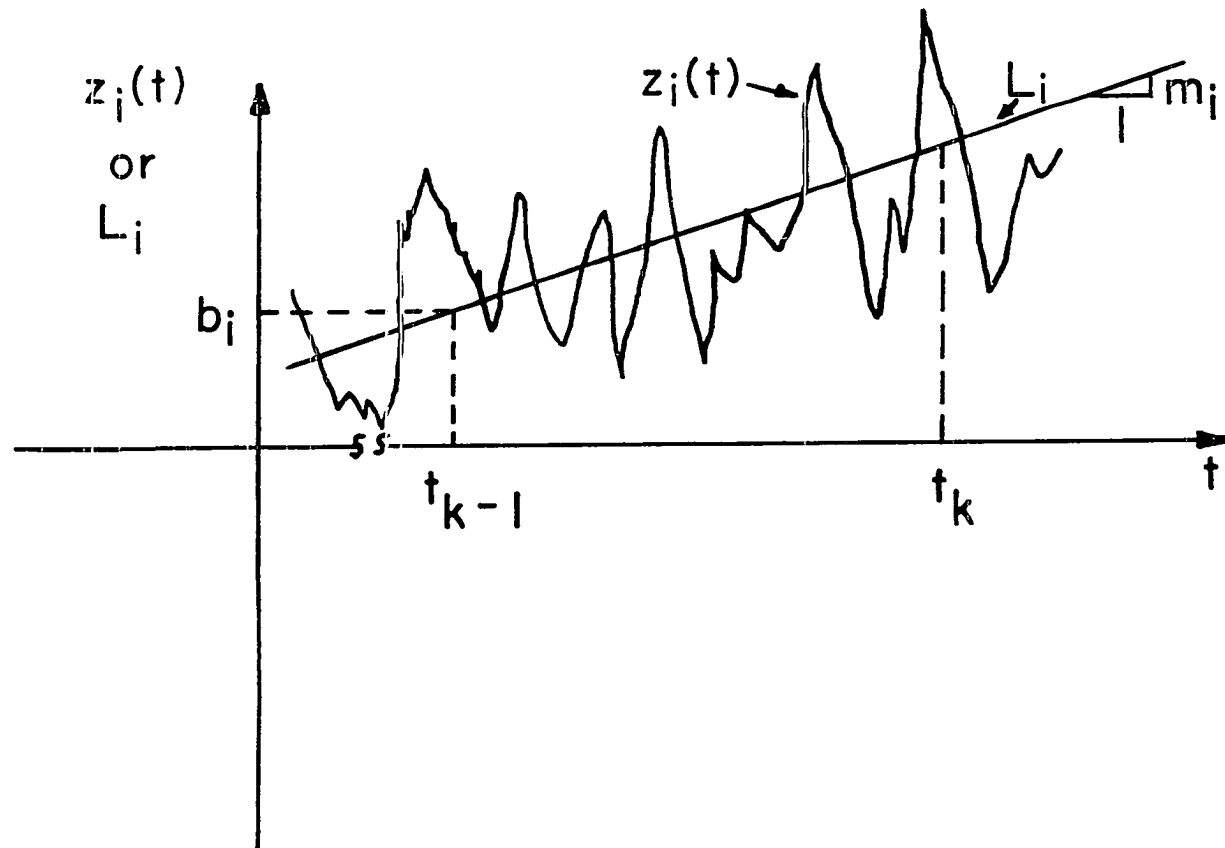
Figure 3.   Linear least square approximation to a noisy measurement

$$F(\underline{m},\underline{b}) \overset{\Delta}{=} \int_{t_{k-1}}^{t_k} [\underline{z}(t) - (t-t_{k-1})\underline{m}-\underline{b}]^T [\underline{z}(t)-(t-t_{k-1})\underline{m}-\underline{b}]dt$$

$$= \int_{t_{k-1}}^{t_k} [z_1(t)-(t-t_{k-1})m_1-b_1]^2 dt + \cdots$$

$$+ \int_{t_{k-1}}^{t_k} [z_i(t)-(t-t_{k-1})m_i-b_i]^2 dt + \cdots$$

$$= minimum \qquad\qquad (4.35)$$

where i varies from 1 to p. Then observe that

$$\frac{\partial F(\underline{m},\underline{b})}{\partial m_i} = \frac{\partial F_i(m_i,b_i)}{\partial m_i} = 0 \qquad\qquad (4.36)$$

is identical to Equation 4.33 and that

$$\frac{\partial F(\underline{m},\underline{b})}{\partial b_i} = \frac{\partial F_i(m_i,b_i)}{\partial b_i} = 0 \qquad\qquad (4.37)$$

is identical to Equation 4.34 since all terms except the one involving m_i and b_i are treated as constants during the partial differentiation process.

Rearranging Equation 4.33 leaves

$$m_i \int_{t_{k-1}}^{t_k} (t-t_{k-1})^2 dt + b_i \int_{t_{k-1}}^{t_k} (t-t_{k-1})dt =$$

$$\int_{t_{k-1}}^{t_k} z_i(t)(t-t_{k-1})dt \qquad\qquad (4.38)$$

Evaluating this by change of variables leads to

$$\frac{\Delta t^3}{3} m_i + \frac{\Delta t^2}{2} b_i = \int_{t_{k-1}}^{t_k} t\, z_i(t)dt - t_{k-1} \int_{t_{k-1}}^{t_k} z_i(t)dt \qquad (4.39)$$

Rearranging Equation 4.34 in a similar manner gives

$$m_i \int_{t_{k-1}}^{t_k} (t-t_{k-1})dt + b_i \int_{t_{k-1}}^{t_k} dt = \int_{t_{k-1}}^{t_k} z_i(t)dt \qquad (4.40)$$

and finally

$$\frac{\Delta t^2}{2} m_i - \Delta t \, b_i = \int_{t_{k-1}}^{t_k} z_i(t)dt \qquad (4.41)$$

Solving Equations 4.39 and 4.41 simultaneously yields

$$b_i = \frac{(6t_{k-1} + 4\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} z_i(t)dt - \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \, z_i(t)dt \qquad (4.42)$$

$$m_i = - \frac{(12t_{k-1} + 6\Delta t)}{\Delta t^3} \int_{t_{k-1}}^{t_k} z_i(t)dt + \frac{12}{\Delta t^3} \int_{t_{k-1}}^{t_k} t \, z_i(t)dt$$
$$(4.43)$$

From Equation 4.30 when $t = t_k$

$$L_i(m_i, t_k, b_i) = \Delta t \, m_i + b_i$$

$$= \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \, z_i(t)dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2}$$

$$\int_{t_{k-1}}^{t_k} z_i(t) \, dt \qquad (4.44)$$

If all measurements of $z_i(t)$ are treated as in Equation 4.44 for $i = 1, 2, \cdots p$ a p-vector $\underline{L}_k$ is formed such that

$$\underline{L}_k = \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \, \underline{z}(t)dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} \underline{z}(t)dt \qquad (4.45)$$

This equation defines the new prefiltering process which consists of sampling the linear least square approximation of

p noisy measurements at $t = t_k$ for processing in a discrete Kalman filter.

It is interesting to compare Equation 4.1 giving the interval averaging process with Equation 4.45. For the case where $\underline{z}(t)$ is a single constant measurement value z_c with zero noise note that

$$z_k = z_c \tag{4.46}$$

and it can be shown that

$$L_k = z_c \tag{4.47}$$

as expected. To demonstrate that the linearized data sampling technique is an improvement over simple data averaging for the cases where measurement values drift, consider a continuous noisy measurement defined as

$$z(t) \overset{\Delta}{=} t + \delta z(t) \tag{4.48}$$

where $\delta z(t)$ is periodic deterministic noise defined

$$\delta z(t) \overset{\Delta}{=} 2 \cos 2\pi t \tag{4.49}$$

If $t_{k-1} = 0$, $t_k = 3$, $\Delta t = 3$ then from Equation 4.1 the averaged value is

$$z_k = \frac{1}{3} \int_0^3 (t + 2 \cos 2\pi t) \, dt$$

$$= 1.5 \tag{4.50}$$

and from Equation 4.45 the linearized sampled value is

$$L_k = \frac{6}{9} \int_0^3 t(t + 2 \cos 2\pi t) dt - \frac{6}{9} \int_0^3 (t + 2 \cos 2\pi t) dt \tag{4.51}$$

$$= 3.0$$

A noisy measurement at $t = 3$ from Equation 4.48 is equal

to 5 while a perfect measurement if possible would be 3. Thus the L_k value is a perfect value and is highly desired over the averaged z_k value. Here, of course, sampling of $z(t)$ would produce a value of 5 which also indicates both smoothing techniques are improvements, in this case, over ordinary sampling.

The equations of the generalized system as shown in Figure 1 are repeated here

$$\underline{\dot{u}}(t) = F(t) \, \underline{u}(t) + \underline{w}(t) \tag{4.2}$$

$$\underline{z}(t) = H(t) \, \underline{u}(t) + \underline{v}(t) \tag{4.3}$$

for the white measurement noise case. Combining Equations 4.3 and 4.45 gives

$$\underline{L}_k = \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \, H(t) \, \underline{u}(t) \, dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2}$$

$$\int_{t_{k-1}}^{t_k} H(t) \, \underline{u}(t) \, dt + \delta\underline{L}_k \tag{4.52}$$

where

$$\delta\underline{L}_k \triangleq \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \, \underline{v}(t) \, dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} \underline{v}(t) \, dt \tag{4.53}$$

New states must be defined to simplify the first two terms on the right-hand side of Equation 4.52 for use in a discrete Kalman filter. Since $\underline{v}(t)$ is an additive white-noise process and is not being treated as a state variable, the latter term of Equation 4.52 remains the modified discrete noise contribution defined $\delta\underline{L}_k$. As with the interval averaging case this method eliminates the problem of infinite variance in a dis-

crete Kalman filter when sampling data containing white measurement noise $\underline{v}(t)$.

The first term of Equation 4.52 can be simplified by applying integration by parts. Consider the integral

$$\int_{t_{k-1}}^{t_k} t \, H(t) \, \underline{u}(t) \, dt$$

and the formula for integration by parts

$$\int_a^b u \, dv = u \, v \, \Big|_a^b - \int_a^b v \, du \qquad (4.54)$$

and let $dv = H(t) \, \underline{u}(t) \, dt$ so $v = \int H(t) \, \underline{u}(t) \, dt$ and let $u = t$ so $du = dt$.

So

$$\int_{t_{k-1}}^{t_k} t \, H(t) \, \underline{u}(t) \, dt = [(t)(\int H(t) \, \underline{u}(t) \, dt)] \, \Big|_{t_{k-1}}^{t_k} - \int_{t_{k-1}}^{t_k}$$

$$(\int H(t) \, \underline{u}(t) \, dt) \, dt \qquad (4.55)$$

Defining states

$$\underline{u}(t) \stackrel{\Delta}{=} \underline{u}(t) \qquad (4.56)$$

$$\underline{y}(t) \stackrel{\Delta}{=} \int H(t) \, \underline{u}(t) \, dt \qquad (4.57)$$

$$\underline{x}(t) \stackrel{\Delta}{=} \int (\int H(t) \, \underline{u}(t) \, dt) \, dt = \int \underline{y}(t) \, dt \qquad (4.58)$$

allows Equation 4.55 to be rewritten in terms of these new states as

$$\int_{t_{k-1}}^{t_k} t \, H(t) \, \underline{u}(t) \, dt = t_k \, \underline{y}(t_k) - t_{k-1} \, \underline{y}(t_{k-1}) - \underline{x}(t_k)$$

$$+ \, \underline{x}(t_{k-1}) \qquad (4.59)$$

Equations 4.56, 4.57, 4.58 and 4.59 can now be combined with

Equation 4.52 to form

$$\underline{L}_k = \frac{6}{\Delta t^2} t_k \underline{y}(t_k) - \frac{6}{\Delta t^2} t_{k-1} \underline{y}(t_{k-1}) - \frac{6}{\Delta t^2} \underline{x}(t_k)$$

$$+ \frac{6}{\Delta t^2} \underline{x}(t_{k-1}) - \frac{6}{\Delta t^2} t_{k-1} \underline{y}(t_k) - \frac{2}{\Delta t} \underline{y}(t_k)$$

$$+ \frac{6}{\Delta t^2} t_{k-1} \underline{y}(t_{k-1}) + \frac{2}{\Delta t} \underline{y}(t_{k-1}) + \delta \underline{L}_k$$

$$= \frac{4}{\Delta t} \underline{y}(t_k) + \frac{2}{\Delta t} \underline{y}(t_{k-1}) - \frac{6}{\Delta t^2} \underline{x}(t_k) + \frac{6}{\Delta t^2} \underline{x}(t_{k-1}) + \delta \underline{L}_k \tag{4.60}$$

This shows that the new equivalent discrete measurement is now a function of only the time interval, the newly defined states at t_k and t_{k-1} and the modified discrete noise contribution.

Differentiated Equations 4.56, 4.57 and 4.58 become

$$\underline{\dot{u}}(t) = F(t) \underline{u}(t) + \underline{w}(t) \tag{4.61}$$

$$\underline{\dot{y}}(t) = H(t) \underline{u}(t) \tag{4.62}$$

$$\underline{\dot{x}}(t) = \underline{y}(t) \tag{4.63}$$

so that the new continuous plant model for the white-noise case is

$$\begin{bmatrix} \underline{\dot{u}}(t) \\ \underline{\dot{y}}(t) \\ \underline{\dot{x}}(t) \end{bmatrix} = \begin{bmatrix} F(t) & 0 & 0 \\ H(t) & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \underline{u}(t) \\ \underline{y}(t) \\ \underline{x}(t) \end{bmatrix} + \begin{bmatrix} \underline{w}(t) \\ 0 \\ 0 \end{bmatrix} \tag{4.64}$$

As before the state transition matrix is defined

$$\frac{d\phi(t,t_{k-1})}{dt} \overset{\Delta}{=} \begin{bmatrix} F(t) & 0 & 0 \\ H(t) & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \phi(t,t_{k-1}) \qquad (4.65)$$

and

$$\phi_{k-1} = \phi(t,t_{k-1}) \Big|_{t=t_k} \qquad (4.66)$$

Observe that if $F(t)$ and $H(t)$ are constant so is the new plant matrix constant. This results from a wise choice of the new states given by Equations 4.56, 4.57, and 4.58. Finally the generalized discrete plant model from Equations 4.64 and 4.66 is

$$\begin{bmatrix} \underline{u}_k \\ \underline{y}_k \\ \underline{x}_k \end{bmatrix} = \phi_{k-1} \begin{bmatrix} \underline{u}_{k-1} \\ \underline{y}_{k-1} \\ \underline{x}_{k-1} \end{bmatrix} + \int_{t_{k-1}}^{t_k} \phi(t_k,\tau) \begin{bmatrix} \underline{w}(\tau) \\ 0 \\ 0 \end{bmatrix} d\tau \qquad (4.67)$$

and the generalized discrete measurement model from Equation 4.60 is

$$\underline{L}_k = [0 \ \frac{4}{\Delta t} \ \frac{-6}{\Delta t^2}] \begin{bmatrix} \underline{u}_k \\ \underline{y}_k \\ \underline{x}_k \end{bmatrix} + [0 \ \frac{2}{\Delta t} \ \frac{6}{\Delta t^2}] \begin{bmatrix} \underline{u}_{k-1} \\ \underline{y}_{k-1} \\ \underline{x}_{k-1} \end{bmatrix} + \delta \underline{L}_k \qquad (4.68)$$

where

$$\delta \underline{L}_k = \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \ \underline{v}(t) \ dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} \underline{v}(t) \ dt \qquad (4.53)$$

The generalized mathematical model given by Equations 4.61, 4.67, and 4.68 is a discrete Kalman filter with delayed

states as observables. It was derived directly from the continuous system Equations 4.2 and 4.3 when sampling optimum-linearized data containing white noise.

D. Sampling Optimum-Linearized Data Containing Colored Noise

The equations of the generalized system with colored measurement noise as shown in Figure 2 are repeated here for convenient reference.

$$\dot{\underline{u}}(t) = F(t) \, \underline{u}(t) + \underline{w}(t) \tag{4.15}$$

$$\underline{z}(t) = H(t) \, \underline{u}(t) + \underline{n}(t) \tag{4.16}$$

$$\underline{n}(t) = A(t) \, \underline{n}(t) + B\underline{v}(t) \tag{4.17}$$

After state vector augmentation, the reformulated system becomes

$$\begin{bmatrix} \dot{\underline{u}}(t) \\ \dot{\underline{n}}(t) \end{bmatrix} = \begin{bmatrix} F(t) & 0 \\ 0 & A(t) \end{bmatrix} \begin{bmatrix} \underline{u}(t) \\ \underline{n}(t) \end{bmatrix} + \begin{bmatrix} \underline{w}(t) \\ B \, \underline{v}(t) \end{bmatrix} \tag{4.18}$$

and

$$\underline{z}(t) = [H(t) \; I] \begin{bmatrix} \underline{u}(t) \\ \underline{n}(t) \end{bmatrix} + 0 \tag{4.19}$$

Applying the optimum-linearized data sampling process

$$\underline{L}_k \triangleq \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \, \underline{z}(t) \, dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} \underline{z}(t) \, dt \tag{4.45}$$

to the reformulated continuous measurement Equation 4.19 gives

$$\underline{L}_k \triangleq \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t\, H(t)\, \underline{u}(t)\, dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2}$$

$$\int_{t_{k-1}}^{t_k} H(t)\, \underline{u}(t)\, dt + \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t\, \underline{n}(t)\, dt - \frac{(6t_{k-1} + 2\Delta t}{\Delta t^2}$$

$$\int_{t_{k-1}}^{t_k} \underline{n}(t)\, dt + \delta\underline{L}_k \tag{4.69}$$

where

$$\delta\underline{L}_k = 0 \tag{4.70}$$

The noise $\underline{n}(t)$ is being treated as a state variable. This allows the measurement model Equation 4.19 to be perfect in augmented form and the corresponding noise term $\delta\underline{L}_k$ is zero. New states must be defined to simplify all terms of Equation 4.69 for use in a discrete Kalman filter. Therefore define states

$$\underline{u}(t) \triangleq \underline{u}(t) \tag{4.71}$$

$$\underline{y}(t) \triangleq \int H(t)\, \underline{u}(t)\, dt \tag{4.72}$$

$$\underline{x}(t) \triangleq \int (\int H(t)\, \underline{u}(t)\, dt)\, dt = \int \underline{y}(t)\, dt \tag{4.73}$$

$$\underline{n}(t) \triangleq \underline{n}(t) \tag{4.74}$$

$$\underline{s}(t) \triangleq \int \underline{n}(t)\, dt \tag{4.75}$$

$$\underline{r}(t) \triangleq \int (\int \underline{n}(t)\, dt)\, dt = \int \underline{s}(t)\, dt \tag{4.76}$$

Observe from Equations 4.55 and 4.71 through 4.76 that

$$\int_{t_{k-1}}^{t_k} t\, H(t)\, \underline{u}(t)\, dt = t_k\, \underline{y}(t_k) - t_{k-1}\, \underline{y}(t_{k-1}) - \underline{x}(t_k)$$

$$+ \underline{x}(t_{k-1}) \tag{4.77}$$

and that

$$\int_{t_{k-1}}^{t_k} t\ \underline{n}(t)\ dt = t_k\ \underline{s}(t_k) - t_{k-1}\ \underline{s}(t_{k-1}) - \underline{r}(t_k) + \underline{r}(t_{k-1})$$

(4.78)

Combining Equations 4.71 through 4.78 with $\underline{L}_k$ given by Equations 4.69 and 4.70 reduces to

$$\underline{L}_k = \frac{4}{\Delta t}\ \underline{y}(t_k) + \frac{2}{\Delta t}\ \underline{y}(t_{k-1}) - \frac{6}{\Delta t^2}\ \underline{x}(t_k) + \frac{6}{\Delta t^2}\ \underline{x}(t_{k-1})$$

(4.79)

$$+ \frac{4}{\Delta t}\ \underline{s}(t_k) + \frac{2}{\Delta t}\ \underline{s}(t_{k-1}) - \frac{6}{\Delta t^2}\ \underline{r}(t_k) + \frac{6}{\Delta t^2}\ \underline{r}(t_{k-1}) + \delta\underline{L}_k$$

where

$$\delta\underline{L}_k = 0$$

(4.70)

Equations 4.71 through 4.76 when differentiated become

$$\dot{\underline{u}}(t) = F(t)\ \underline{u}(t) + \underline{w}(t)$$

(4.80)

$$\dot{\underline{y}}(t) = H(t)\ \underline{u}(t)$$

(4.81)

$$\dot{\underline{x}}(t) = \underline{y}(t)$$

(4.82)

$$\dot{\underline{n}}(t) = A(t)\ \underline{n}(t) + B\ \underline{v}(t)$$

(4.83)

$$\dot{\underline{s}}(t) = \underline{n}(t)$$

(4.84)

$$\dot{\underline{r}}(t) = \underline{s}(t)$$

(4.85)

which implies that the new continuous plant model for the colored noise case is

$$\begin{bmatrix} \dot{\underline{u}}(t) \\ \dot{\underline{y}}(t) \\ \dot{\underline{x}}(t) \\ \dot{\underline{n}}(t) \\ \dot{\underline{s}}(t) \\ \dot{\underline{r}}(t) \end{bmatrix} = \begin{bmatrix} F(t) & 0 & 0 & 0 & 0 & 0 \\ H(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A(t) & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \underline{u}(t) \\ \underline{y}(t) \\ \underline{x}(t) \\ \underline{n}(t) \\ \underline{s}(t) \\ \underline{r}(t) \end{bmatrix} + \begin{bmatrix} \underline{w}(t) \\ 0 \\ 0 \\ B\ \underline{v}(t) \\ 0 \\ 0 \end{bmatrix}$$

(4.86)

Defining the state transition matrix as before where

$$\frac{d}{dt}\phi(t,t_{k-1}) \triangleq \begin{bmatrix} F(t) & 0 & 0 & 0 & 0 & 0 \\ H(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A(t) & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}\phi(t,t_{k-1}) \qquad (4.87)$$

the generalized discrete plant model from Equation 4.86 is

$$\begin{bmatrix} \underline{u}_k \\ \underline{y}_k \\ \underline{x}_k \\ \underline{n}_k \\ \underline{s}_k \\ \underline{r}_k \end{bmatrix} = \phi_{k-1} \begin{bmatrix} \underline{u}_{k-1} \\ \underline{y}_{k-1} \\ \underline{x}_{k-1} \\ \underline{n}_{k-1} \\ \underline{s}_{k-1} \\ \underline{r}_{k-1} \end{bmatrix} + \int_{t_{k-1}}^{t_k} \phi(t_k,\tau) \begin{bmatrix} \underline{w}(\tau) \\ 0 \\ 0 \\ B\,\underline{v}(\tau) \\ 0 \\ 0 \end{bmatrix} d\tau \qquad (4.88)$$

and the generalized discrete measurement model from Equations 4.79 and 4.70 is

$$\underline{L}_k = [0 \ \frac{4}{\Delta t} \ \frac{-6}{\Delta t^2} \ 0 \ \frac{4}{\Delta t} \ \frac{-6}{\Delta t^2}] \begin{bmatrix} \underline{u}_k \\ \underline{y}_k \\ \underline{x}_k \\ \underline{n}_k \\ \underline{s}_k \\ \underline{r}_k \end{bmatrix} + [0 \ \frac{2}{\Delta t} \ \frac{6}{\Delta t^2} \ 0 \ \frac{2}{\Delta t} \ \frac{6}{\Delta t^2}] \begin{bmatrix} \underline{u}_{k-1} \\ \underline{y}_{k-1} \\ \underline{x}_{k-1} \\ \underline{n}_{k-1} \\ \underline{s}_{k-1} \\ \underline{r}_{k-1} \end{bmatrix} + \delta\underline{L}_k \qquad (4.89)$$

where

$$\delta\underline{L}_k = 0 \qquad (4.70)$$

The generalized mathematical model Equations 4.70, 4.88 and 4.89 comprise a discrete Kalman filter with delayed states as observables. It was derived directly from the continuous system Equations 4.15, 4.16, and 4.17 when sampling optimum-linearized data containing colored noise.

V. AN EXAMPLE WITH WHITE MEASUREMENT NOISE

An example with one additive white-noise input, or plant driving function, and a single output corrupted by additive white noise is considered in this chapter. The block diagram for this continuous system is shown in Figure 4. The system equations for this example corresponding to Equations 3.1 and 3.2 for the plant and measurement models are respectively

$$\dot{u}(t) = (0)\ u(t) + w(t) \tag{5.1}$$

$$z(t) = (1)\ u(t) + v(t) \tag{5.2}$$

The plant driving function $w(t)$ and measurement noise $v(t)$ are white-noise processes with statistics

$$E\{w(t)\} = 0 \qquad \text{for all } t \tag{5.3}$$

$$E\{w(t_1)w(t_2)\} = \alpha\delta(t_1 - t_2) \tag{5.4}$$

and

$$E\{v(t)\} = 0 \qquad \text{for all } t \tag{5.5}$$

$$E\{v(t_1)\ v(t_2)\} = \beta\delta(t_1 - t_2) \tag{5.6}$$

where α and β are arbitrary scalar constants.

Notice that for this one input-one output example the quantities $u(t)$, $w(t)$, $z(t)$, and $v(t)$ are not vectors. This type of system is chosen so that the mathematical solution and interpretation of the results would be less complicated than those for the more general type of problem treated in the previous chapter. Also observe that the plant and measurement models are continuous; therefore, evaluations can be performed by either discrete or continuous filter techniques.

$$\phi_W(\tau) = a\delta(\tau)$$
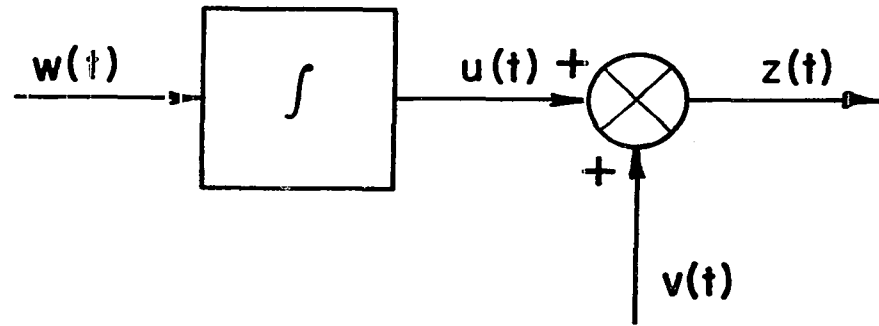
$$a = \text{constant}$$

$$\phi_V(\tau) = \beta\delta(\tau)$$

$$\beta = \text{constant}$$

Figure 4.   Block diagram for example with white measurement noise

The usual discrete Kalman filter cannot be applied due to the white measurement noise which causes an infinite variance of a sampled measurement value. The continuous Kalman-Bucy filter yielding optimum continuous results provides a basis for comparison of the modified discrete filter results.

This completes the specification and general discussion of the problem. In the following sections the interval-averaging, linearized-sampling, and Kalman-Bucy filter techniques are applied to the system of this example.

A. Interval-Averaging Filter

The recursive filter equations developed here are based on the generalized system equations as given in Section A of Chapter IV where the interval-averaging process is defined by Equation 4.1. Comparing the generalized continuous system Equations 4.2 and 4.3 with the system Equations 5.1 and 5.2 of the example shows that

$$F(t) = 0 \qquad\qquad (5.7)$$

$$H(t) = 1 \qquad\qquad (5.8)$$

The new continuous plant model Equation 4.9 becomes

$$\begin{bmatrix} \dot{u}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ 0 \end{bmatrix} \qquad (5.9)$$

Since the new plant matrix is a constant coefficient matrix, then by Equations 4.10 and 4.11

$$\phi_{k-1} = \phi(\Delta t) = \mathscr{L}^{-1}\left\{ sI - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}^{-1}$$

$$= \mathscr{L}^{-1}\begin{bmatrix} s & 0 \\ -1 & s \end{bmatrix}^{-1}$$

$$= \mathscr{L}^{-1}\begin{bmatrix} 1/s & 0 \\ 1/s^2 & 1/s \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \tag{5.10}$$

Equation 4.12 can be rewritten as

$$\begin{bmatrix} u_k \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} u_{k-1} \\ y_{k-1} \end{bmatrix} + \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 & 0 \\ (t_k-\tau) & 1 \end{bmatrix} \begin{bmatrix} w(\tau) \\ 0 \end{bmatrix} d\tau \tag{5.11}$$

which becomes after some simplifications the new discrete

plant as described by

$$\begin{bmatrix} u_k \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} u_{k-1} \\ y_{k-1} \end{bmatrix} + \begin{bmatrix} \int_{t_{k-1}}^{t_k} w(\tau)d\tau \\ \int_{t_{k-1}}^{t_k} (t_k-\tau) \, w(\tau)d\tau \end{bmatrix} \tag{5.12}$$

The new discrete measurement model is given by Equations 4.13

and 4.14 and after a few changes is

$$z_k = [0 \; \tfrac{1}{\Delta t}] \begin{bmatrix} u_k \\ y_k \end{bmatrix} + [0 \; \tfrac{-1}{\Delta t}] \begin{bmatrix} u_{k-1} \\ y_{k-1} \end{bmatrix} + \delta z_k \tag{5.13}$$

where

$$\delta z_k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} v(t) \, dt \tag{5.14}$$

This new discrete system defined by Equations 5.12, 5.13, and 5.14 is identical in form to the Kalman filter with delayed states as given in Chapter III where

$$\underline{x}_k = \phi_{k-1}\,\underline{x}_{k-1} + \underline{w}_{k-1} \qquad (3.9)$$

$$\underline{z}_k = M_k\,\underline{x}_k + N_k\,\underline{x}_{k-1} + \underline{v}_k \qquad (3.23)$$

Thus by direct correspondence

$$\underline{x}_k = \begin{bmatrix} u_k \\ y_k \end{bmatrix} \qquad (5.15)$$

$$\phi_{k-1} = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \qquad (5.10)$$

$$\underline{w}_{k-1} = \begin{bmatrix} \displaystyle\int_{t_{k-1}}^{t_k} w(\tau)\,d\tau \\[2em] \displaystyle\int_{t_{k-1}}^{t_k} (t_k-\tau)w(\tau)\,d\tau \end{bmatrix} \qquad (5.16)$$

$$\underline{z}_k = z_k \qquad (5.17)$$

$$M_k = [\,0 \;\; \tfrac{1}{\Delta t}\,] \qquad (5.18)$$

$$N_k = [\,0 \;\; \tfrac{-1}{\Delta t}\,] = -M_k \qquad (5.19)$$

$$\underline{v}_k = \delta z_k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} v(t)\,dt \qquad (5.20)$$

From Equations 5.6 and 5.20 and the definition of the measurement noise covariance matrix

$$V_k \overset{\Delta}{=} E\{\underline{v}_k \ \underline{v}_k^T\} = \frac{1}{\Delta t^2} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} E\{v(t_1) \ v(t_2)\} \ dt_2 \ dt_1$$

$$= \frac{1}{\Delta t^2} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \beta\delta(t_1 - t_2) \ dt_2 \ dt_1$$

$$= \frac{\beta}{\Delta t} \qquad\qquad (5.21)$$

Using the definition of the plant noise covariance matrix,

$$H_{k-1} = E\{\underline{w}_{k-1} \ \underline{w}_{k-1}^T\}$$

and Equations 5.4 and 5.16, leads to the equations

$$H_{k-1}(1,1) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} E\{w(\tau_1) \ w^T(\tau_2)\} \ d\tau_2 \ d\tau_1$$

$$= \alpha \int_{t_{k-1}}^{t_k} d\tau_1$$

$$= \alpha \ \Delta t$$

$$H_{k-1}(1,2) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_2) E\{w(\tau_1) w^T(\tau_2)\} \ d\tau_2 \ d\tau_1$$

$$= \alpha \int_{t_{k-1}}^{t_k} (t_k - \tau_1) \ d\tau_1$$

$$= \alpha \ \frac{\Delta t^2}{2}$$

$$H_{k-1}(2,1) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_1) E\{w(\tau_1)w^T(\tau_2)\} d\tau_2 \, d\tau_1$$

$$= \alpha \int_{t_{k-1}}^{t_k} (t_k - \tau_1) \, d\tau_1$$

$$= \alpha \frac{\Delta t^2}{2}$$

$$H_{k-1}(2,2) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_1) E\{w(\tau_1)w^T(\tau_2)\}(t_k - \tau_2) \, d\tau_2 \, d\tau_1$$

$$= \alpha \int_{t_{k-1}}^{t_k} (t_k - \tau_1)^2 \, d\tau_1$$

$$= \alpha \frac{\Delta t^3}{3}$$

and combining these equations leads to

$$H_{k-1} = \begin{bmatrix} \alpha \, \Delta t & \alpha \, \dfrac{\Delta t^2}{2} \\[2ex] \alpha \, \dfrac{\Delta t^2}{2} & \alpha \, \dfrac{\Delta t^3}{3} \end{bmatrix} \tag{5.22}$$

All necessary quantities have now been determined for the application of the delayed state recursive Kalman filtering Equations 3.21, 3.26, 3.27, and 3.28. Calculations can be performed after selecting values of the discrete time interval and values of the white noise amplitudes α and β.

B. Linearized-Sampling Filter

Generalized filter equations with delayed states were derived in Section C of Chapter IV for cases where optimum-linearized continuous measurements containing white noise were sampled. These equations are now applied directly to this example. As in the previous section

$$F(t) = 0 \tag{5.7}$$

$$H(t) = 1 \tag{5.8}$$

The new continuous plant model given by Equation 4.64 after substitution of Equations 5.7 and 5.8 is

$$\begin{bmatrix} \dot{u}(t) \\ \dot{y}(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ 0 \\ 0 \end{bmatrix} \tag{5.23}$$

The plant matrix is a constant coefficient matrix; therefore, Equations 4.65 and 4.66 imply that

$$\phi_{k-1} = \phi(\Delta t)$$

$$= \mathcal{L}^{-1} \left\{ sI - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}^{-1}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} s & 0 & 0 \\ -1 & s & 0 \\ 0 & -1 & s \end{bmatrix}^{-1}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} 1/s & 0 & 0 \\ 1/s^2 & 1/s & 0 \\ 1/s^3 & 1/s^2 & 1/s \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \Delta t & 1 & 0 \\ \Delta t^2/2 & \Delta t & 1 \end{bmatrix} \tag{5.24}$$

Rewriting Equation 4.67 using Equation 5.24 yields the new discrete plant model

$$\begin{bmatrix} u_k \\ y_k \\ x_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \Delta t & 1 & 0 \\ \Delta t^2/2 & \Delta t & 1 \end{bmatrix} \begin{bmatrix} u_{k-1} \\ y_{k-1} \\ x_{k-1} \end{bmatrix} + \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 & 0 & 0 \\ (t_k-\tau) & 1 & 0 \\ (t_k-\tau)^2/2 & (t_k-\tau) & 1 \end{bmatrix} \begin{bmatrix} w(\tau) \\ 0 \\ 0 \end{bmatrix} d\tau$$

Simplifying this equation produces

$$\begin{bmatrix} u_k \\ y_k \\ x_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \Delta t & 1 & 0 \\ \Delta t^2/2 & \Delta t & 1 \end{bmatrix} \begin{bmatrix} u_{k-1} \\ y_{k-1} \\ x_{k-1} \end{bmatrix} + \begin{bmatrix} \int_{t_{k-1}}^{t_k} w(\tau) \, d\tau \\ \int_{t_{k-1}}^{t_k} (t_k-\tau) \, w(\tau) \, d\tau \\ \int_{t_{k-1}}^{t_k} (t_k-\tau)^2/2 \, w(\tau) \, d\tau \end{bmatrix} \tag{5.25}$$

The new discrete measurement model given by Equations 4.53 and 4.68 with some minor changes is

$$L_k = [0 \quad 4/\Delta t \quad -6/\Delta t^2] \begin{bmatrix} u_k \\ y_k \\ x_k \end{bmatrix} + [0 \quad 2/\Delta t \quad 6/\Delta t^2] \begin{bmatrix} u_{k-1} \\ y_{k-1} \\ x_{k-1} \end{bmatrix} + \delta L_k \qquad (5.26)$$

where

$$\delta L_k = \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \ v(t) \ dt - \frac{(6t_{k-1} + 2\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} v(t) \ dt \qquad (5.27)$$

Noting the direct relationship of the discrete system defined by Equations 5.25, 5.26 and 5.27 with the delayed state Kalman filter equations

$$\underline{x}_k = \phi_{k-1} \ \underline{x}_{k-1} + \underline{w}_{k-1} \qquad (3.9)$$

$$\underline{z}_k = M_k \ \underline{x}_k + N_k \ \underline{x}_{k-1} + \underline{v}_k \qquad (3.23)$$

leads to the following observations

$$\underline{x}_k = \begin{bmatrix} u_k \\ y_k \\ x_k \end{bmatrix} \qquad (5.28)$$

$$\phi_{k-1} = \begin{bmatrix} 1 & 0 & 0 \\ \Delta t & 1 & 0 \\ \Delta t^2/2 & \Delta t & 1 \end{bmatrix} \qquad (5.24)$$

$$
\underline{w}_{k-1} = \begin{bmatrix} \int_{t_{k-1}}^{t_k} & & w(\tau) \ d\tau \\[2em] \int_{t_{k-1}}^{t_k} (t_k-\tau) & & w(\tau) \ d\tau \\[2em] \int_{t_{k-1}}^{t_k} (t_k-\tau)^2/2 & & w(\tau) \ d\tau \end{bmatrix} \qquad (5.29)
$$

$$
z_k = L_k
$$

$$
M_k = [0 \quad 4/\Delta t \quad -6/\Delta t^2]
$$

$$
N_k = [0 \quad 2/\Delta t \quad 6/\Delta t^2]
$$

$$
\underline{v}_k = \delta L_k
$$

$$
= \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} t \ v(t) \ dt - \frac{(6 \ t_{k-1} + 2\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} v(t) \ dt \qquad (5.30)
$$

Several additional terms must now be evaluated. From Equations 5.6 and 5.30

$$
V_k = E\{\underline{v}_k \ \underline{v}_k^T\}
$$

$$
= \frac{36}{\Delta t^4} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} t_1 \ E\{v(t_1) \ v(t_2)\} \ t_2 \ dt_2 \ dt_1
$$

$$
+ \frac{(6 \ t_{k-1} + 2\Delta t)^2}{\Delta t^4} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} E\{v(t_1) \ v(t_2)\} dt_2 \ dt_1
$$

$$- \frac{(6)(6\ t_{k-1} + 2\Delta t)}{\Delta t^4} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} t_1\ E\{v(t_1)\ v(t_2)\} dt_2\ dt_1$$

$$- \frac{(6)(6\ t_{k-1} + 2\Delta t)}{\Delta t^4} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} E\{v(t_1)\ v(t_2)\} t_2\ dt_2\ dt_1$$

$$= \frac{12\beta}{\Delta t^4} (t_k^3 - t_{k-1}^3)$$

$$+ \frac{\beta(6t_{k-1} + 2\Delta t)^2}{\Delta t^4} (\Delta t)$$

$$- \frac{3\beta(6t_{k-1} \div 2\Delta t)}{\Delta t^4} (t_k^2 - t_{k-1}^2)$$

$$- \frac{3\beta(6t_{k-1} + 2\Delta t)}{\Delta t^4} (t_k^2 - t_{k-1}^2)$$

$$= \frac{4\beta}{\Delta t} \hspace{6cm} (5.31)$$

Likewise using Equations 5.4 and 5.29, the definition

$$H_{k-1} \overset{\Delta}{=} E\{\underline{w}_{k-1}\ \underline{w}_{k-1}^T\}$$

and considering each element separately leads to

$$H_{k-1}(1,1) = \alpha \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \delta(\tau_1 - \tau_2)\ d\tau_2\ d\tau_1$$

$$= \alpha \int_{t_{k-1}}^{t_k} d\tau_1$$

$$= \alpha \Delta t$$

$$H_{k-1}(2,2) = \alpha \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_1)(t_k - \tau_2)\delta(\tau_1 - \tau_2) \; d\tau_2 \; d\tau_1$$

$$= \alpha \int_{t_{k-1}}^{t_k} (t_k - \tau_1)^2 \; d\tau_1$$

$$= \alpha \frac{\Delta t^3}{3}$$

$$H_{k-1}(3,3) = \alpha \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{(t_k - \tau_1)^2}{2} \frac{(t_k - \tau_2)^2}{2} \delta(\tau_1 - \tau_2) \; d\tau_2 \; d\tau_1$$

$$= \frac{\alpha}{4} \int_{t_{k-1}}^{t_k} (t_k - \tau_1)^4 \; d\tau_1$$

$$= \alpha \frac{\Delta t^5}{20}$$

$$H_{k-1}(1,2) = H_{k-1}(2,1)$$

$$= \alpha \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_2)\delta(\tau_1 - \tau_2) \; d\tau_2 \; d\tau_1$$

$$= \alpha \int_{t_{k-1}}^{t_k} (t_k - \tau_1) \; d\tau_1$$

$$= \alpha \frac{\Delta t^2}{2}$$

$$H_{k-1}(1,3) = H_{k-1}(3,1)$$

$$= \alpha \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{(t_k - \tau_2)^2}{2} \delta(\tau_1 - \tau_2) \, d\tau_2 \, d\tau_1$$

$$= \frac{\alpha}{2} \int_{t_{k-1}}^{t_k} (t_k - \tau_1)^2 \, d\tau_1$$

$$= \alpha \frac{\Delta t^3}{6}$$

$$H_{k-1}(2,3) = H_{k-1}(3,2)$$

$$= \alpha \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_1) \left(\frac{t_k - \tau_2}{2}\right)^2 \delta(\tau_1 - \tau_2) \, d\tau_2 \, d\tau_1$$

$$= \frac{\alpha}{2} \int_{t_{k-1}}^{t_k} (t_k - \tau_1)^3 \, d\tau_1$$

$$= \alpha \frac{\Delta t^4}{8}$$

Combining these results to a more compact form leads to the equation

$$H_{k-1} = \alpha \begin{bmatrix} \Delta t & \Delta t^2/2 & \Delta t^3/6 \\ \Delta t^2/2 & \Delta t^3/3 & \Delta t^4/8 \\ \Delta t^3/6 & \Delta t^4/8 & \Delta t^5/20 \end{bmatrix} \qquad (5.32)$$

The delayed state recursive Kalman filtering equations have now been completely formed for the linearized-sampling filter and only the numerical evaluation remains to be performed.

C. Continuous Kalman-Bucy Filter

This example as shown in Figure 4 is a continuous system. Therefore Kalman-Bucy filter equations exist for this system. In particular, a continuous optimal gain matrix and the continuous error covariance matrix for this optimal gain can be determined in accordance with the methods outlines in Appendix A. The steady-state value of the continuous error covariance for the state u(t) in the example is a lower bound which is approached only from above by discrete filters as Δt approaches zero.

Formulating the example in terms of Equations 10.1 and 10.2 results in

$$F(t) = 0 \qquad (5.33)$$

$$M(t) = 1 \qquad (5.34)$$

$$Q(t) = \alpha \qquad (5.35)$$

$$R(t) = \beta \qquad (5.36)$$

From Equation 10.24 the set of equivalent equations, used to solve the Ricatti equation for the error covariance matrix $P(t|t)$ becomes

$$\begin{bmatrix} \dot{Y}(t) \\ \dot{Z}(t) \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \frac{1}{\beta} & 0 \end{bmatrix} \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} \qquad (5.37)$$

The transition matrix associated with this equation is

$$\phi(t,t_o) = \mathcal{L}^{-1} \left\{ sI - \begin{bmatrix} 0 & \alpha \\ \frac{1}{\beta} & 0 \end{bmatrix} \right\}^{-1}$$

$$= \mathscr{L}^{-1} \begin{bmatrix} s & -\alpha \\ -\dfrac{1}{\beta} & s \end{bmatrix}^{-1}$$

$$= \mathscr{L}^{-1} \begin{bmatrix} \dfrac{s}{s^2 - (\sqrt{\frac{\alpha}{\beta}})^2} & \dfrac{\alpha}{s^2 - (\sqrt{\frac{\alpha}{\beta}})^2} \\ \dfrac{1/\beta}{s^2 - (\sqrt{\frac{\alpha}{\beta}})^2} & \dfrac{s}{s^2 - (\sqrt{\frac{\alpha}{\beta}})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \cosh \sqrt{\tfrac{\alpha}{\beta}}\,(t-t_0) & \sqrt{\alpha\beta}\,\sinh \sqrt{\tfrac{\alpha}{\beta}}\,(t-t_0) \\ \dfrac{1}{\sqrt{\alpha\beta}}\,\sinh \sqrt{\tfrac{\alpha}{\beta}}\,(t-t_0) & \cosh \sqrt{\tfrac{\alpha}{\beta}}\,(t-t_0) \end{bmatrix} \qquad (5.38)$$

Using Equation 5.38 in 10.30 if $a = \sqrt{\alpha\beta}$ and $b = \sqrt{\alpha/\beta}$ produces

$$P(t|t) = Y(t)\,Z^{-1}(t)$$

$$= \frac{P_0 \cosh b\,(t-t_0) + a \sinh b\,(t-t_0)}{P_0\,\dfrac{1}{a}\sinh b\,(t-t_0) + \cosh b\,(t-t_0)}$$

$$= \frac{P_0\left[\dfrac{e^{b(t-t_0)} + e^{-b(t-t_0)}}{2}\right] + a\left[\dfrac{e^{b(t-t_0)} - e^{-b(t-t_0)}}{2}\right]}{\dfrac{P_0}{a}\left[\dfrac{e^{b(t-t_0)} - e^{-b(t-t_0)}}{2}\right] + \left[\dfrac{e^{b(t-t_0)} + e^{-b(t-t_0)}}{2}\right]}$$

$$= \frac{P_0[1 + e^{-2b(t-t_0)}] + a[1 - e^{-2b(t-t_0)}]}{\dfrac{P_0}{a}[1 - e^{-2b(t-t_0)}] + [1 + e^{-2b(t-t_0)}]}$$

$$= \frac{a[1 + \frac{P_0}{a}] - a[1 - \frac{P_0}{a}] \, e^{-2b(t-t_0)}}{[1 + \frac{P_0}{a}] + [1 - \frac{P_0}{a}] \, e^{-2b(t-t_0)}} \tag{5.38}$$

where

$$a = \sqrt{\alpha\beta} \tag{5.39}$$

and where

$$b = \sqrt{\frac{\alpha}{\beta}} \tag{5.40}$$

The steady-state value of the continuous error-covariance for the state u(t) is defined as

$$P_\infty \overset{\Delta}{=} \lim_{t \to \infty} P(t \,|t) \tag{5.41}$$

Combining the convenient form of Equation 5.38 with Equations 5.39 and 5.41 reduces to

$$P_\infty = \frac{a(1 + \frac{P_0}{a}) - 0}{(1 + \frac{P_0}{a}) + 0}$$

$$= a$$

$$= \sqrt{\alpha\beta} \tag{5.42}$$

D. Results

Computation of the interval averaging and linearized-sampling processes from recursive equations derived in preceding Sections A and B was accomplished by computer. The delayed-state Kalman filter equations were programmed using Fortran IV language and processed on the Iowa State University IBM-360, Model 65 computer. After choosing α, β, Δt, k = 1

and $P_0 = 0$, iterations on k were performed until the a poster-
iori estimation error covariance, $P_k(1,1)$, of the state u(t)
reached a steady state value defined as $P_{ss}(1,1)$. In all
cases this value was the (1,1) element of the a posteriori
estimation error-covariance matrix since the state u(t) was
always the first element of the state vector. It was found
necessary to perform all computations in double precision in
order to prevent Q_k in Equation 3.26 from going to zero. This
was essential since the inverse of Q_k is required in Equation
3.27. Notice that $P_0 = 0$ implies that the initial error-
covariance matrix was set equal to zero. The parameters, α
and β, of plant and measurement white noise were chosen in
each case to demonstrate the relative effects of noise ampli-
tude on the estimation error covariance as the discrete time
interval varied.

The computed steady-state a posteriori estimation error-
covariance value, $P_{ss}(1,1)$, of state u(t) is shown plotted
versus the discrete time interval in Figures 5 through 9
where the selected values of α and β are as indicated on each
figure. The lower bound of the continuous case steady-state
error-covariance value, P_∞, is also illustrated in each
figure based on the Kalman-Bucy filter in Section C.

For this simple example it was possible to mathematically
process the kth iteration in general terms obtaining the dif-
ference equation of the error-covariance value of state u(t)

Figure 5. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for white measurement noise example of Figure 4

Figure 6. Computed steady-state a posteriori estimation error-covariance
$P_{ss}(1,1)$ of state $u(t)$ for white measurement noise example of
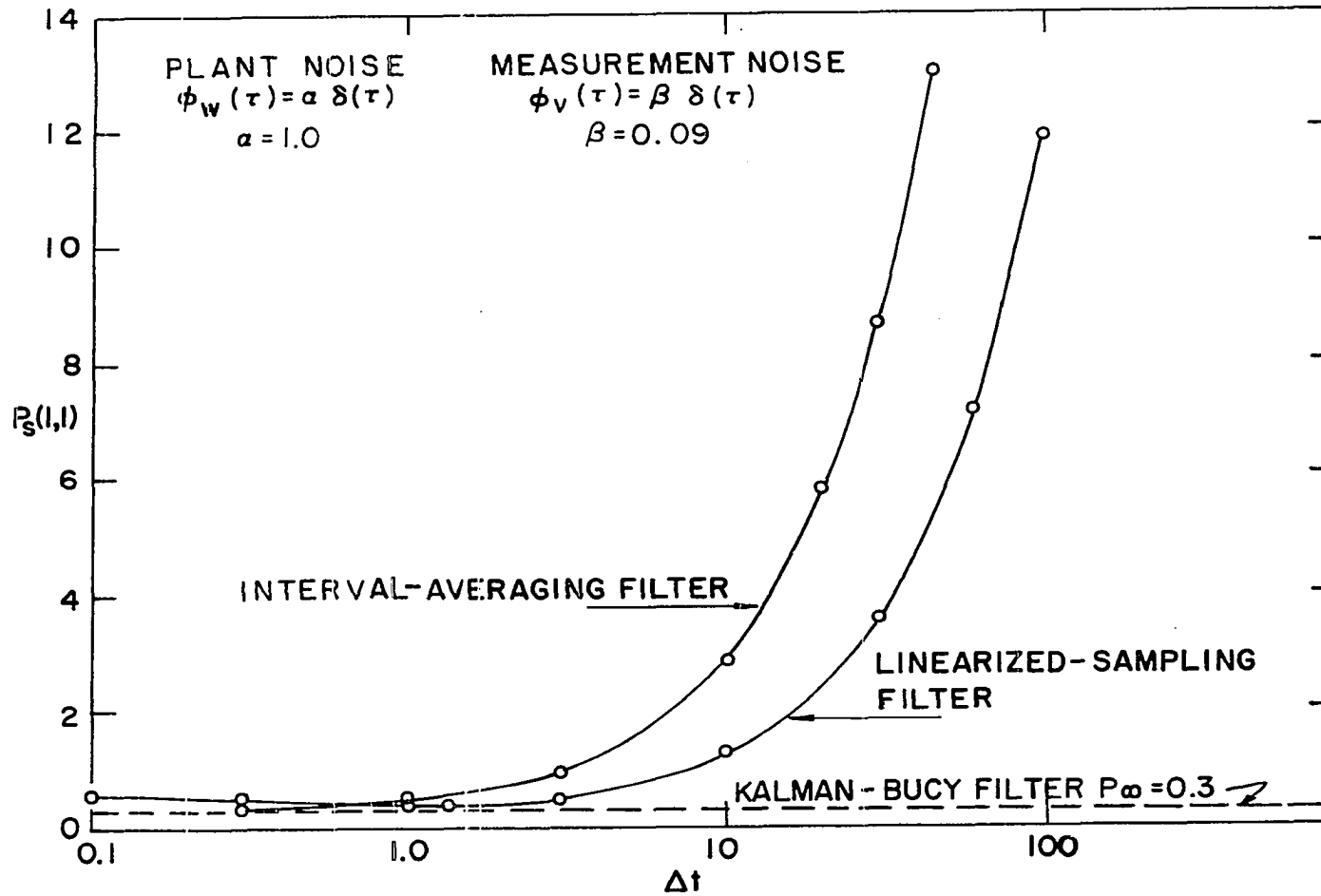Figure 4

Figure 7. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for white measurement noise example of Figure 4

Figure 8. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for white measurement noise example of Figure 4
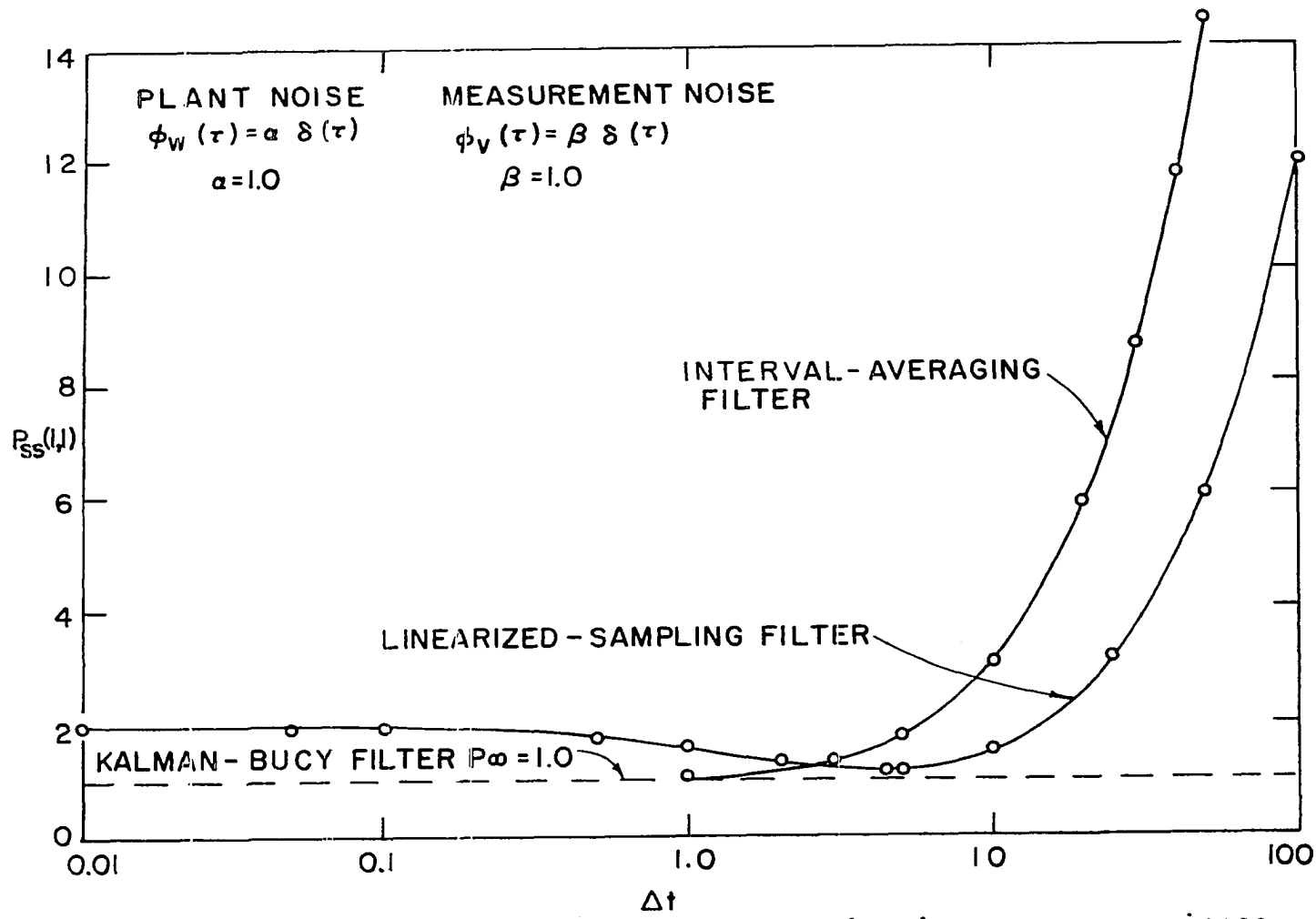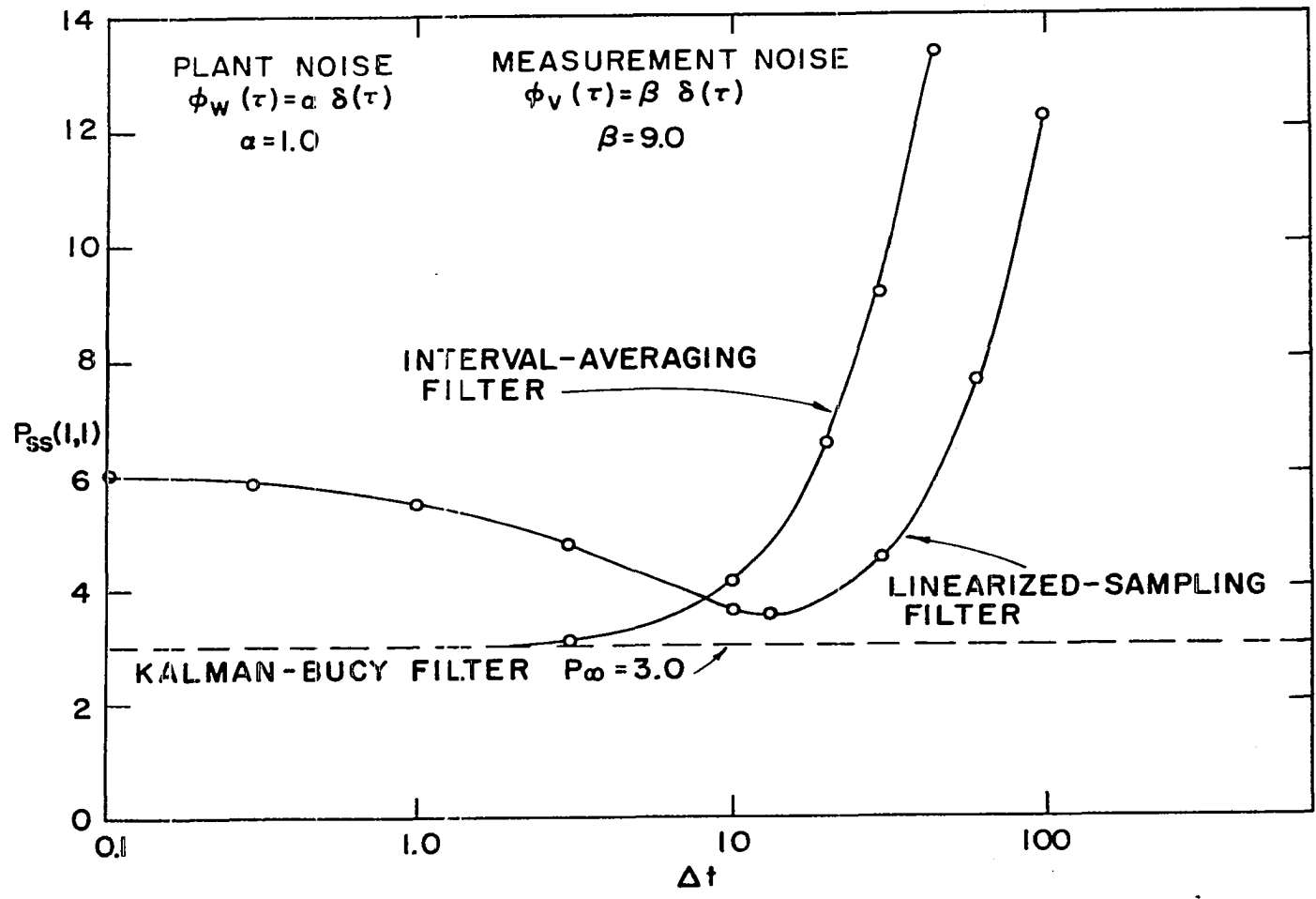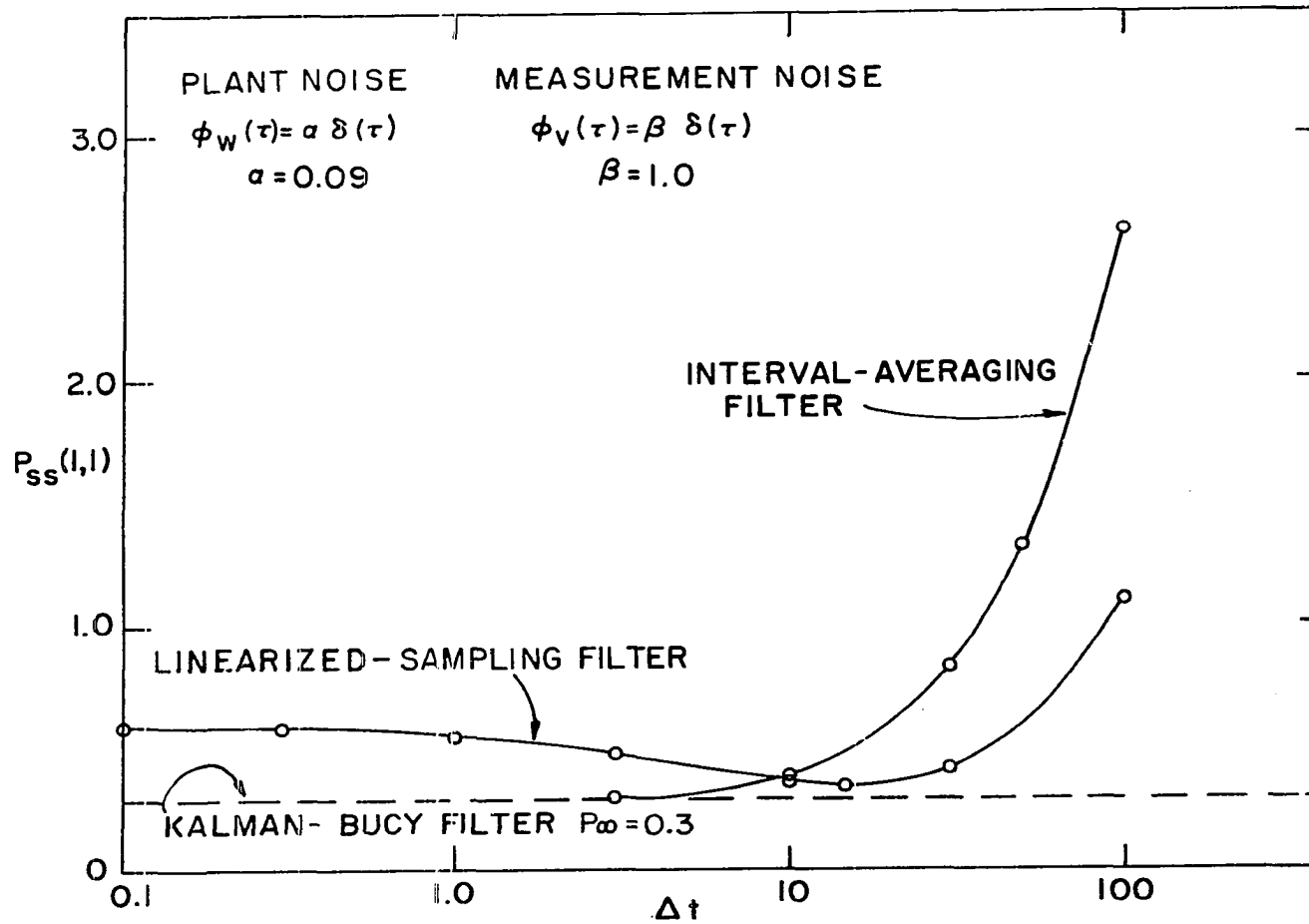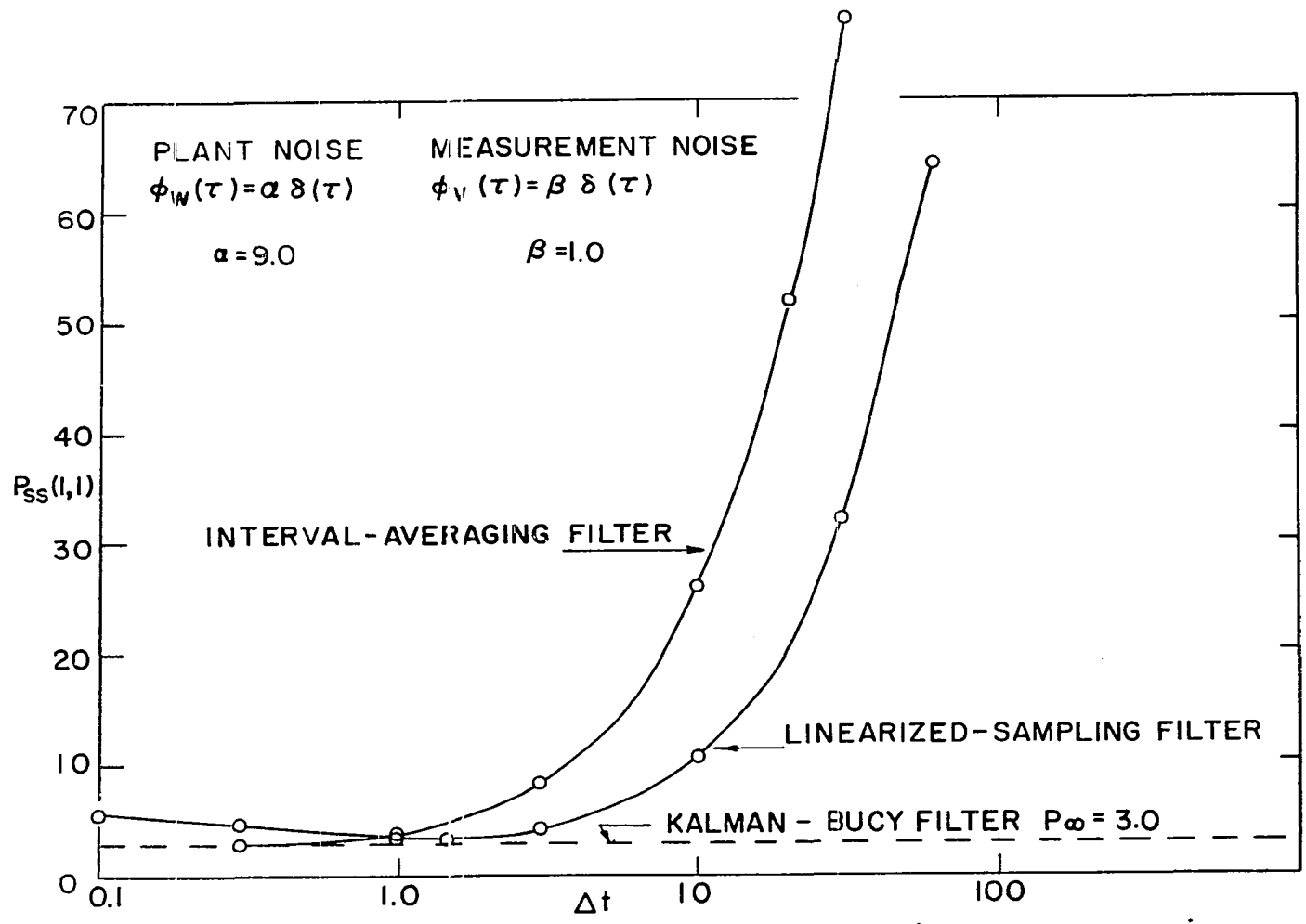
Figure 9. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state u(t) for white measurement noise example of Figure 4

at t_k strictly as a function of Δt and of the error-covariance value at t_{k-1}. This was done for both the interval-averaging and linearized-sampling filters to provide a check of all computations. Steady-state conditions imply that the error-covariance values of u(t) at t_k and t_{k-1} should be equal when t is very large, i.e., $P_k(1,1) = P_{k-1}(1,1) = P_{ss}(1,1)$. This condition was imposed on the difference equation with the following results. The steady-state error-covariance value of u(t) for the interval-averaging filter is

$$P_{ss}(1,1) = \alpha\sqrt{\frac{\beta}{\alpha} + 0.08333333\Delta t^2} \qquad (5.43)$$

and for the linearized-sampling filter is

$$P_{ss}(1,1) = \frac{\alpha}{2}\left[\sqrt{\frac{23}{15}\Delta t^2 + 16\frac{\beta}{\alpha}} - \Delta t\right] \qquad (5.44)$$

The values of $P_{ss}(1,1)$ for Equations 5.43 and 5.44 did agree exactly with all iterative computer results.

The steady-state error-covariance value approaches the Kalman-Bucy P_∞ as Δt becomes small for each case of the integral-averaging filter. The linearized-sampling result decreases towards P_∞ as Δt is reduced but then increases and in the limit approaches a value greater than P_∞. The cause of this increase in steady-state error as Δt decreases will be discussed in following paragraphs in detail. Figures 5 through 9 clearly indicate, for the discrete time interval less than one, that the interval-averaging technique offers the best discrete filter. For the discrete time interval

greater than one the linearized-sampling process is best. In all cases this process yields a minimum $P_{ss}(1,1)$ which does not occur as Δt approaches zero but which does occur at a Δt greater than one. The fact that this minimum value occurs at increasingly larger discrete intervals as the measurement noise amplitude increases relative to the plant noise amplitude is verified in Figures 5, 6, and 7. Even though equivalent measurement noise covariance, V_k, for linearized-sampling is four times the value for interval-averaging, the quantities M_k, N_k and H_{k-1} combine to increase Q_k and to hold b_k approximately constant for values of Δt greater than 1. From Equation 3.28 it is obvious that the term $b_k \, Q_k \, b_k^T$ will increase with the ultimate result being a reduction in the steady-state value of $P_k(1,1)$. The results show as expected that increasing measurement noise amplitude also increases $P_{ss}(1,1)$ for all filters.

The increase in the linearized-sampling $P_{ss}(1,1)$ above P_∞ for very small Δt occurs when the linear least square fit approximates the noise rather than the trend of the signal. Consider a noisy measurement

$$z(t) = 2 + \delta z(t) \tag{5.45}$$

where the deterministic noise is

$$\delta z(t) = \sin \frac{(t - t_{k-1}) 2\pi}{\Delta t} \tag{5.46}$$

If $t_{k-1} = 0$ and $\Delta t = 1$ then using the interval-averaging

process, Equation 4.1 produces

$$z_k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} z(t) \, dt$$

$$= \int_0^1 (2 + \sin 2\pi t) \, dt$$

$$= 2 \tag{5.47}$$

and the linearized-sampling process, Equation 4.45, produces

$$L_k = \frac{6}{\Delta t^2} \int_{t_{k-1}}^{t_k} (t) \, z(t) \, dt - \frac{(6 t_{k-1} + 2\Delta t)}{\Delta t^2} \int_{t_{k-1}}^{t_k} z(t) \, dt$$

$$= 6 \int_0^1 (t)(2 + \sin 2\pi t) \, dt - 2 \int_0^1 (2 + \sin 2\pi t) \, dt$$

$$= 1.045 \tag{5.48}$$

The averaging process yields a perfect discrete value. The linearized-sampling process approximates the noise and produces a poor sampled value L_k at t_k. Note that this problem occurs when the sampling frequency approaches the noise frequency which is usually the case as Δt tends to zero.

VI. AN EXAMPLE WITH MARKOV MEASUREMENT NOISE

The system with white measurement noise which was ana-
lyzed in Chapter V demonstrated that the linearized-sampling
filter is an improvement over the interval-averaging filter
for discrete time intervals greater than one. Of course this
improvement has only been shown for white measurement noise.
White measurement noise implies zero time correlation which
may be the case if discrete measurements are taken at widely
spaced time intervals. Typical continuous measurement proc-
esses are much more likely to have a noise which exhibits
correlation at different instants of time i.e., colored noise.
Thus, in order to verify that this improvement does in fact
exist for a realistic or practical system, an example with
one additive white-noise input and a single output corrupted
by Markov noise is considered in this chapter.

The block diagram for this system is shown in Figure 10.
The system equations for this example corresponding to Equa-
tions 4.15 and 4.16 for the plant and measurement models are
respectively

$$\dot{u}(t) = (0) \ u(t) + w(t) \tag{6.1}$$

$$z(t) = u(t) + n(t) \tag{6.2}$$

The zero-mean plant white noise $w(t)$ and the zero-mean meas-
urement Markov noise $n(t)$ are processes with statistics

$$E\{w(t)\} = 0 \quad \text{for all } t \tag{6.3}$$

$$E\{w(t_1) \ w(t_2)\} = \alpha\delta(t_1 - t_2) \tag{6.4}$$

$$\phi_w(\tau) = \alpha \; \delta(\tau)$$

$$\alpha = \text{CONSTANT}$$

$$\phi_n(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$\sigma^2 = \text{CONSTANT}$$
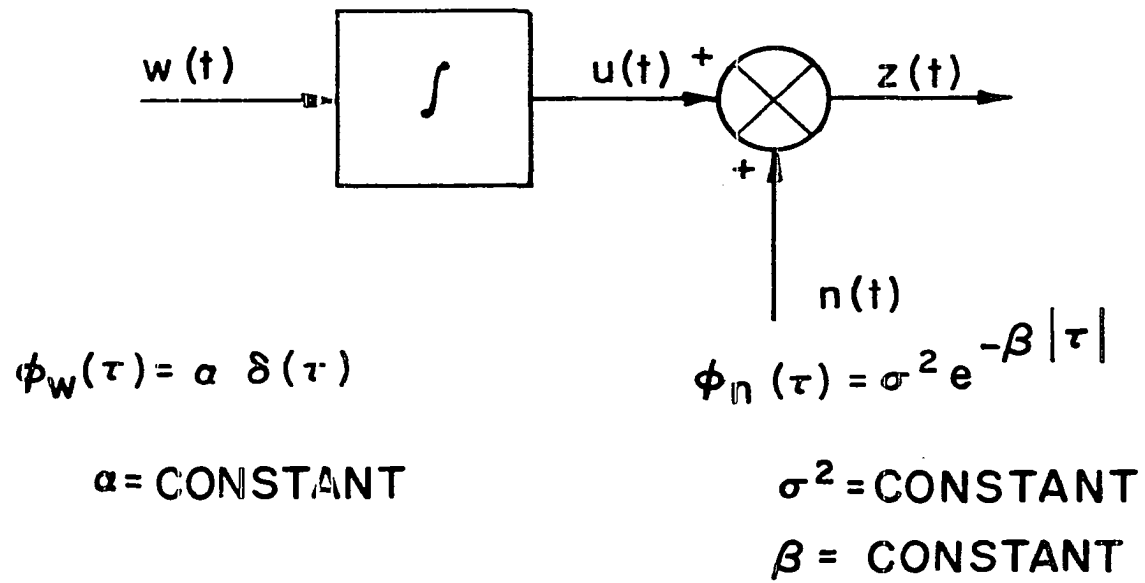
$$\beta = \text{CONSTANT}$$

Figure 10. Block diagram for example with Markov measurement noise

and

$$E\{n(t)\} = 0 \qquad \text{for all } t \qquad (6.5)$$

$$E\{n(t_1)\ n(t_2)\} = \sigma^2\ e^{-\beta|t_1 - t_2|} \qquad (6.6)$$

where α, β, and σ^2 are arbitrary scalar constants. The quantities $u(t)$, $w(t)$, $z(t)$, and $n(t)$ are not vectors since this example consists of only one input and one output.

This model can be reformulated as illustrated in Figure 11 by using a shaping filter with an input $v(t)$ of additive white noise

$$E\{v(t)\} = 0 \qquad \text{for all } t \qquad (6.7)$$

$$E\{v(t_1)\ v(t_2)\} = \delta(t_1 - t_2) \qquad (6.8)$$

and an output $n(t)$ of Markov noise defined by Equations 6.5 and 6.6. The statistical character of $n(t)$ is defined by a shaping filter equation similar to Equation 4.17 as

$$\dot{n}(t) = -\beta n(t) + \sqrt{2\sigma^2\beta}\ v(t) \qquad (6.9)$$

State vector augmentation yields the reformulated system

$$\begin{bmatrix} \dot{u}(t) \\ \dot{n}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} u(t) \\ n(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ \sqrt{2\sigma^2\beta}\ v(t) \end{bmatrix} \qquad (6.10)$$

$$z(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ n(t) \end{bmatrix} + 0 \qquad (6.11)$$

where it is assumed that white noise processes $w(t)$ and $v(t)$ are uncorrelated. The augmented system measurements are observed to be perfect from Equation 6.11.

As with the previous example, filter analysis can be

$$\phi_W(\tau) = a\ \delta(\tau)$$
$$a = \text{constant}$$
$$\phi_n(\tau) = \sigma^2 e^{-\beta|\tau|}$$
$$\sigma^2 = \text{constant}$$
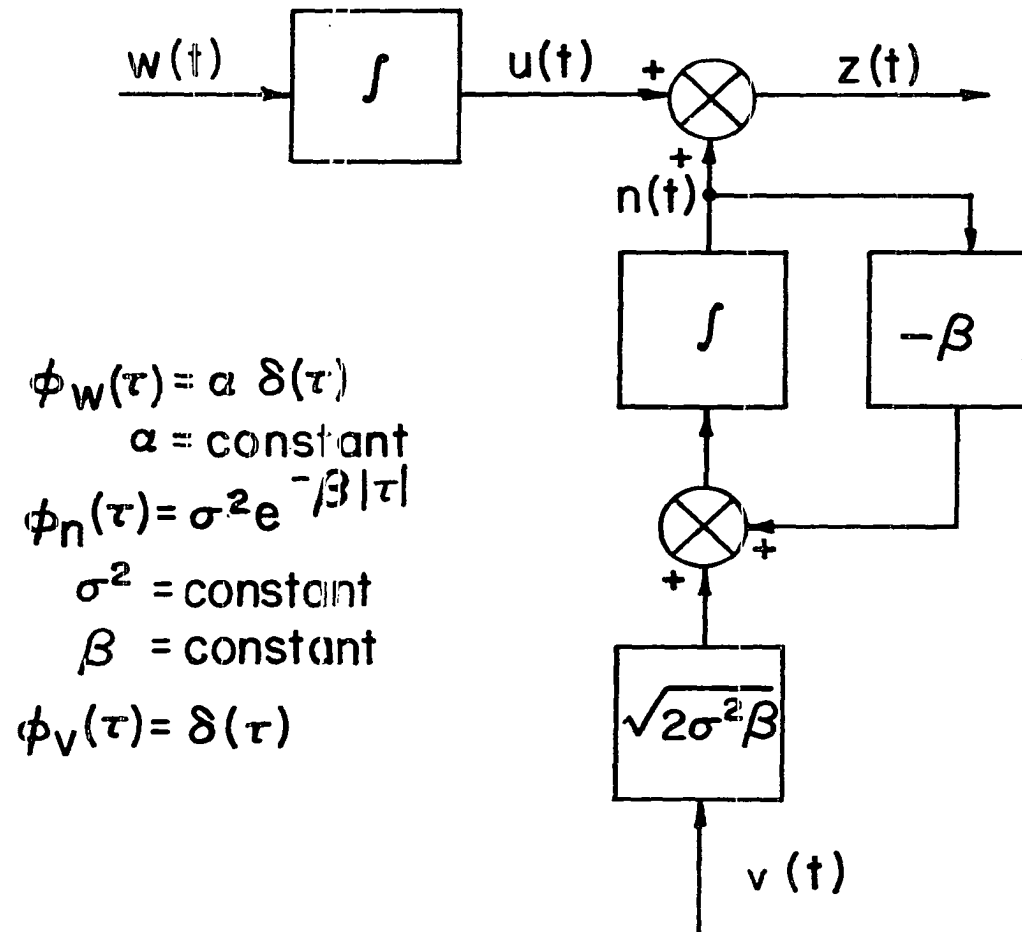$$\beta = \text{constant}$$
$$\phi_V(\tau) = \delta(\tau)$$

Figure 11. Block diagram for example with Markov measurement noise after addition of shaping filter

performed either by discrete or continuous methods because the reformulated plant and measurement models in Equations 6.10 and 6.11 are continuous. The discrete Kalman filter can be used in this example since sampled continuous measurements containing Markov noise produce only finite variances. Both the continuous Kalman-Bucy filter yielding the lower bound of steady-state estimation error covariance and the discrete Kalman filter provide a basis for comparison of the modified discrete filter results.

This completes the derivation and general discussion of the reformulated system after state vector augmentation. In the remainder of this chapter the discrete Kalman, interval-averaging, linearized-sampling, and Kalman-Bucy filter techniques are applied to the system of this example.

A.   Discrete Kalman Filter

The recursive Kalman filter equations for this example will be developed based on the generalized equations derived in Section B of Chapter III. Comparing the reformulated system, Equations 6.10 and 6.11 with the continuous model Equations 3.1 and 3.2 show that

$$\bar{F}(t) = \bar{F}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\beta \end{bmatrix} \tag{6.12}$$

$$\underline{w}(t) = \begin{bmatrix} w(t) \\ \sqrt{2\sigma^2\beta} & v(t) \end{bmatrix} \tag{6.13}$$

$$M(t) = [1 \quad 1] \tag{6.14}$$

$$\underline{v}(t) = 0 \tag{6.15}$$

Since F is a constant, the transition matrix may be determined from Equation 3.8 as

$$\phi_{k-1} = \phi(\Delta t)$$

$$= \mathcal{L}^{-1}\{sI-F\}^{-1}$$

$$= \mathcal{L}^{-1}\left\{ sI - \begin{bmatrix} 0 & 0 \\ 0 & -\beta \end{bmatrix} \right\}^{-1}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} s & 0 \\ 0 & s-\beta \end{bmatrix}^{-1}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \dfrac{1}{s} & 0 \\ 0 & \dfrac{1}{s-\beta} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & e^{-\beta\Delta t} \end{bmatrix} \tag{6.16}$$

Using this result gives the discrete model from Equations 3.9, 3.10, and 3.11 as

$$\begin{bmatrix} u_k \\ n_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\beta\Delta t} \end{bmatrix} \begin{bmatrix} u_{k-1} \\ n_{k-1} \end{bmatrix} + \begin{bmatrix} \displaystyle\int_{t_{k-1}}^{t_k} w(\tau)\,d\tau \\ \displaystyle\int_{t_{k-1}}^{t_k} e^{-\beta(t_k-\tau)} \sqrt{2\sigma^2\beta}\; v(\tau)\,d\tau \end{bmatrix} \tag{6.17}$$

$$z_k = [1 \quad 1] \begin{bmatrix} u_k \\ n_k \end{bmatrix} + 0 \tag{6.18}$$

where

$$\underline{w}_{k-1} = \begin{bmatrix} \displaystyle\int_{t_{k-1}}^{t_k} w(\tau) \, d\tau \\[20pt] \displaystyle\int_{t_{k-1}}^{t_k} e^{-\beta(t_k-\tau)} \sqrt{2\sigma^2\beta} \; v(\tau) \, d\tau \end{bmatrix} \tag{6.19}$$

$$\underline{v}_k = 0 \tag{6.20}$$

$$M_k = [1 \quad 1] \tag{6.21}$$

and where the noiseless measurement z_k is the sampled value of $z(t)$ at time t_k. Several additional quantities must now be evaluated. Notice that the measurement noise covariance

$$V_k \overset{\Delta}{=} E\{\underline{v}_k \; \underline{v}_k^T\}$$

$$= 0 \tag{6.22}$$

is finite for this example. The plant noise covariance matrix is defined as

$$H_{k-1} \overset{\Delta}{=} E\{\underline{w}_{k-1} \; \underline{w}_{k-1}^T\}$$

which gives the following elements from Equation 6.19.

$$H_{k-1}(1,1) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} E\{w(\tau_1)w(\tau_2)\} d\tau_2 \, d\tau_1$$

$$H_{k-1}(1,2) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \sqrt{2\sigma^2\beta} \; e^{-\beta(t_k-\tau_2)} E\{w(\tau_1)v(\tau_2)\; d\tau_2 \; d\tau_1$$

$$H_{k-1}(2,1) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \sqrt{2\sigma^2\beta} e^{-\beta(t_k-\tau_1)} E\{v(\tau_1)w(\tau_2)\}d\tau_2 \; d\tau_1$$

$$H_{k-1}(2,2) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} 2\sigma^2\beta \; e^{-\beta(t_k-\tau_1)} e^{-\beta(t_k-\tau_2)}$$

$$E\{v(\tau_1) \; v(\tau_2)\} \; d\tau_2 \; d\tau_1$$

Using Equations 6.4 and 6.8 in these equations and recombining them into a matrix equation reduces to

$$H_{k-1} = \begin{bmatrix} \alpha\Delta t & 0 \\ 0 & \sigma^2(1-e^{-2\beta\Delta t}) \end{bmatrix} \qquad (6.23)$$

after recalling that white noise processes w(t) and v(t) are uncorrelated. The Kalman discrete filter equations

$$P_k^* = \phi_{k-1} \; P_{k-1} \; \phi_{k-1}^T + H_{k-1} \qquad (3.21)$$

$$K_k = P_k^* \; M_k^T \; (M_k \; P_k^* \; M_k^T + V_k)^{-1} \qquad (3.18)$$

$$P_k = P_k^* - K_k \; (M_k \; P_k^* \; M_k^T + V_k) \; K_k^T \qquad (3.19)$$

can now be computed since all necessary coefficients have been determined.

B. Interval-Averaging Filter

The interval-averaging process defined by Equation 4.1 will be applied to this example using the generalized equations derived in Section B of Chapter IV. Comparing the generalized reformulated system, Equations 4.18 and 4.19 with the reformulated system Equations 6.10 and 6.11 implies that

$$F(t) = 0 \qquad\qquad (6.24)$$

$$A(t) = -\beta \qquad\qquad (6.25)$$

$$B = \sqrt{2\sigma^2\beta} \qquad\qquad (6.26)$$

$$H(t) = 1 \qquad\qquad (6.27)$$

The interval-averaging process creates the new continuous model from Equation 4.26 which is

$$
\begin{bmatrix} \dot{u}(t) \\ \dot{n}(t) \\ \dot{y}(t) \\ \dot{s}(t) \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} u(t) \\ n(t) \\ y(t) \\ s(t) \end{bmatrix}
+
\begin{bmatrix} w(t) \\ \sqrt{2\sigma^2\beta}\, v(t) \\ 0 \\ 0 \end{bmatrix}
\qquad (6.28)
$$

Since the new plant matrix is a constant matrix, then by Equations 3.8, 4.11, and 4.27

$$
\phi_{k-1} = \mathcal{L}^{-1} \left\{ sI - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} \right\}
$$

$$
= \mathcal{L}^{-1}
\begin{bmatrix}
\dfrac{1}{s} & 0 & 0 & 0 \\[2mm]
0 & \dfrac{1}{s+\beta} & 0 & 0 \\[2mm]
\dfrac{1}{s^2} & 0 & \dfrac{1}{s} & 0 \\[2mm]
0 & \dfrac{1}{s(s+\beta)} & 0 & \dfrac{1}{s}
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
1 & 0 & 0 & 0 \\[2mm]
0 & e^{-\beta\Delta t} & 0 & 0 \\[2mm]
\Delta t & 0 & 1 & 0 \\[2mm]
0 & \dfrac{(1-e^{-\beta\Delta t})}{\beta} & 0 & 1
\end{bmatrix}
\tag{6.29}
$$

Rewriting Equation 4.28, it becomes after reduction the new discrete plant model given by the equation

$$
\begin{bmatrix}
u_k \\[4mm]
n_k \\[4mm]
y_k \\[4mm]
s_k
\end{bmatrix}
= \phi_{k-1}
\begin{bmatrix}
u_{k-1} \\[4mm]
n_{k-1} \\[4mm]
y_{k-1} \\[4mm]
s_{k-1}
\end{bmatrix}
+
\begin{bmatrix}
\displaystyle\int_{t_{k-1}}^{t_k} w(\tau)\, d\tau \\[4mm]
\displaystyle\int_{t_{k-1}}^{t_k} e^{-\beta(t_k-\tau)}\sqrt{2\sigma^2\beta}\; v(\tau)\, d\tau \\[4mm]
\displaystyle\int_{t_{k-1}}^{t_k} (t_k-\tau)\, w(\tau)\, d\tau \\[4mm]
\displaystyle\int_{t_{k-1}}^{t_k} [1-e^{-\beta(t_k-\tau)}]\, \dfrac{\sqrt{2\sigma^2\beta}}{\beta}\, v(\tau)\, d\tau
\end{bmatrix}
\tag{6.30}
$$

The new discrete measurement model from Equation 4.29 is

$$z_k = [0 \ 0 \ \frac{1}{\Delta t} \ \frac{1}{\Delta t}] \begin{bmatrix} u_k \\ n_k \\ y_k \\ s_k \end{bmatrix} + [0 \ 0 \ \frac{-1}{\Delta t} \ \frac{-1}{\Delta t}] \begin{bmatrix} u_{k-1} \\ n_{k-1} \\ y_{k-1} \\ s_{k-1} \end{bmatrix} + 0 \qquad (6.31)$$

Comparing this new mathematical system model Equations 6.30 and 6.31 with the delay-state Kalman filter equations where

$$\underline{x}_k = \phi_{k-1} \ \underline{x}_{k-1} + \underline{w}_{k-1} \qquad (3.9)$$

$$\underline{z}_k = M_k \ \underline{x}_{k-1} + N_k \ \underline{x}_{k-1} + \underline{v}_k \qquad (3.23)$$

by direct correspondence gives

$$\underline{x}_k = \begin{bmatrix} u_k \\ n_k \\ y_k \\ s_k \end{bmatrix} \qquad (6.32)$$

$$\phi_{k-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\beta \Delta t} & 0 & 0 \\ \Delta t & 0 & 1 & 0 \\ 0 & \frac{(1-e^{-\beta \Delta t})}{\beta} & 0 & 1 \end{bmatrix} \qquad (6.33)$$

$$\underline{w}_{k-1} = \begin{bmatrix} \int_{t_{k-1}}^{t_k} w(\tau)\ d\tau \\[2ex] \int_{t_{k-1}}^{t_k} \sqrt{2\sigma^2\beta}\ e^{-\beta(t_k-\tau)}\ v(\tau)\ d\tau \\[2ex] \int_{t_{k-1}}^{t_k} (t_k-\tau)\ w(\tau)\ d\tau \\[2ex] \int_{t_{k-1}}^{t_k} \frac{\sqrt{2\sigma^2\beta}}{\beta}\ [1-e^{-\beta(t_k-\tau)}]v(\tau)\ d\tau \end{bmatrix} \tag{6.34}$$

$$\underline{z}_k = z_k \tag{6.35}$$

$$M_k = [0 \quad 0 \quad \frac{1}{\Delta t} \quad \frac{1}{\Delta t}] \tag{6.36}$$

$$N_k = [0 \quad 0 \quad \frac{-1}{\Delta t} \quad \frac{-1}{\Delta t}] = -M_k \tag{6.37}$$

$$\underline{v}_k = 0 \tag{6.38}$$

The measurement noise covariance matrix is

$$V_k \overset{\Delta}{=} E\{\underline{v}_k\ \underline{v}_k^T\}$$

$$= 0 \tag{6.39}$$

and the plant noise covariance matrix is given by the definition

$$H_{k-1} \overset{\Delta}{=} E\{\underline{w}_{k-1}\ \underline{w}_{k-1}^T\}$$

which becomes

$$H_{k-1}(1,1) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} E\{w(\tau_1)\ w(\tau_2)\}\ d\tau_2\ d\tau_1$$

$$= \alpha\Delta t \tag{6.40}$$

$$H_{k-1}(2,2) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} 2\sigma^2 \beta e^{-\beta(t_k-\tau_1)} e^{-\beta(t_k-\tau_2)}$$

$$E\{v(\tau_1) \ v(\tau_2)\} \ d\tau_2 \ d\tau_1$$

$$= \sigma^2 (1 - e^{-2\beta\Delta t}) \qquad\qquad (6.41)$$

$$H_{k-1}(3,3) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k-\tau_1)(t_k-\tau_2) E\{w(\tau_1)w(\tau_2)\} \ d\tau_2 \ d\tau_1$$

$$= \alpha \frac{\Delta t^3}{3} \qquad\qquad (6.42)$$

$$H_{k-1}(4,4) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{2\sigma^2}{\beta} [1-e^{-\beta(t_k-\tau_1)}][1-e^{-\beta(t_k-\tau_2)}]$$

$$E\{v(\tau_1) \ v(\tau_2)\} \ d\tau_2 \ d\tau_1$$

$$= \frac{2\sigma^2}{\beta} [\Delta t + \frac{1}{2\beta} - \frac{1}{2\beta} (2-e^{-\beta\Delta t})^2] \qquad\qquad (6.43)$$

$$H_{k-1}(1,3) = H_{k-1}(3,1)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k-\tau_2) \ E\{w(\tau_1) \ w(\tau_2)\} \ d\tau_2 \ d\tau_1$$

$$= \alpha \frac{\Delta t^2}{2} \qquad\qquad (6.44)$$

$$H_{k-1}(2,4) = H_{k-1}(4,2)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} 2\sigma^2 \, e^{-\beta(t_k-\tau_1)} [1-e^{-\beta(t_k-\tau_2)}]$$

$$E\{v(\tau_1) \, v(\tau_2)\} \, d\tau_2 \, d\tau_1$$

$$= \frac{\sigma^2}{\beta} (1 - e^{-\beta\Delta t})^2 \qquad (6.45)$$

The H_{k-1} elements (1,2), (2,1), (1,4), (4,1), (2,3), (3,2), (3,4), and (4,3) are all zero since $w(t)$ and $v(t)$ are uncorrelated.

The delayed-state Kalman filtering Equations 3.21, 3.26, and 3.28 may now be applied as all required quantities have been evaluated. Computations will be performed for various discrete time intervals after preselecting the noise parameters α, σ^2, and β.

C. Linearized-Sampling Filter

The linearized-sampling process defined by Equation 4.45 will be evaluated for this example using the generalized equations derived in Section D of Chapter IV. From the previous section Equations 6.24, 6.25, 6.26, and 6.27 still apply for the reformulated system and the new continuous plant model from Equation 4.86 is

$$
\begin{bmatrix} \dot{u}(t) \\ \dot{y}(t) \\ \dot{x}(t) \\ \dot{n}(t) \\ \dot{s}(t) \\ \dot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \\ x(t) \\ n(t) \\ s(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ 0 \\ 0 \\ \sqrt{2\sigma^2\beta}\,v(t) \\ 0 \\ 0 \end{bmatrix} \qquad (6.46)
$$

The state transition matrix for this constant plant matrix from Equation 4.87 is

$$
\phi_{k-1} = \mathcal{L}^{-1} \left\{ sI - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right\}^{-1}
$$

$$
= \mathcal{L}^{-1} \begin{bmatrix} \dfrac{1}{s} & 0 & 0 & 0 & 0 & 0 \\[2ex] \dfrac{1}{s^2} & \dfrac{1}{s} & 0 & 0 & 0 & 0 \\[2ex] \dfrac{1}{s^3} & \dfrac{1}{s^2} & \dfrac{1}{s} & 0 & 0 & 0 \\[2ex] 0 & 0 & 0 & \dfrac{1}{s+\beta} & 0 & 0 \\[2ex] 0 & 0 & 0 & \dfrac{1}{s(s+\beta)} & \dfrac{1}{s} & 0 \\[2ex] 0 & 0 & 0 & \dfrac{1}{s^2(s+\beta)} & \dfrac{1}{s^2} & \dfrac{1}{s} \end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\Delta t & 1 & 0 & 0 & 0 & 0 \\
\dfrac{\Delta t^2}{2} & \Delta t & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{-\beta \Delta t} & 0 & 0 \\
0 & 0 & 0 & \dfrac{(1-e^{-\beta \Delta t})}{\beta} & 1 & 0 \\
0 & 0 & 0 & \dfrac{(\beta \Delta t - 1 + e^{-\beta \Delta t})}{\beta^2} & \Delta t & 1
\end{bmatrix}
\tag{6.47}
$$

Rewriting Equation 4.88 and simplifying the latter term gives the new discrete plant model as

$$
\begin{bmatrix} u_k \\ y_k \\ x_k \\ n_k \\ s_k \\ r_k \end{bmatrix}
= \phi_{k-1}
\begin{bmatrix} u_{k-1} \\ y_{k-1} \\ x_{k-1} \\ n_{k-1} \\ s_{k-1} \\ r_{k-1} \end{bmatrix}
+
\begin{bmatrix}
\displaystyle \int_{t_{k-1}}^{t_k} w(\tau)\, d\tau \\[2em]
\displaystyle \int_{t_{k-1}}^{t_k} (t_k - \tau)\, w(\tau)\, d\tau \\[2em]
\displaystyle \int_{t_{k-1}}^{t_k} \frac{(t_k - \tau)^2}{2}\, w(\tau)\, d\tau \\[2em]
\displaystyle \int_{t_{k-1}}^{t_k} e^{-\beta(t_k - \tau)} \sqrt{2\sigma^2 \beta}\; v(\tau)\, d\tau \\[2em]
\displaystyle \int_{t_{k-1}}^{t_k} \frac{[1 - e^{-\beta(t_k - \tau)}]}{\beta} \sqrt{2\sigma^2 \beta}\; v(\tau)\, d\tau \\[2em]
\displaystyle \int_{t_{k-1}}^{t_k} \frac{[\beta(t_k - \tau) - 1 + e^{-\beta(t_k - \tau)}]}{\beta^2} \sqrt{2\sigma^2 \beta}\; v(\tau)\, d\tau
\end{bmatrix}
\tag{6.48}
$$

where ϕ_{k-1} is given by Equation 6.47. The new discrete measurement model is obtained from Equation 4.89 as

$$L_k = [0 \ \frac{4}{\Delta t} \ \frac{-6}{\Delta t^2} \ 0 \ \frac{4}{\Delta t} \ \frac{-6}{\Delta t^2}] \begin{bmatrix} u_k \\ y_k \\ x_k \\ n_k \\ s_k \\ r_k \end{bmatrix} + [0 \ \frac{2}{\Delta t} \ \frac{6}{\Delta t^2} \ 0 \ \frac{2}{\Delta t} \ \frac{6}{\Delta t^2}] \begin{bmatrix} u_{k-1} \\ y_{k-1} \\ x_{k-1} \\ n_{k-1} \\ s_{k-1} \\ r_{k-1} \end{bmatrix} + 0$$

(6.49)

Direct correspondence of Equations 6.48 and 6.49 with the delayed-state Kalman filter equations

$$\underline{x}_k = \phi_{k-1} \ \underline{x}_{k-1} + \underline{w}_{k-1}$$

(3.9)

$$\underline{z}_k = M_k \ \underline{x}_k + N_k \ \underline{x}_{k-1} + \underline{v}_k$$

(3.23)

provides the following relationships

$$\underline{x}_k = \begin{bmatrix} u_k \\ y_k \\ x_k \\ n_k \\ s_k \\ r_k \end{bmatrix}$$

(6.50)

The state transition matrix ϕ_{k-1} is given by Equation 6.47.

$$\underline{w}_{k-1} = \begin{bmatrix} \int_{t_{k-1}}^{t_k} w(\tau) \, d\tau \\[2ex] \int_{t_{k-1}}^{t_k} (t_k - \tau) \, w(\tau) \, d\tau \\[2ex] \int_{t_{k-1}}^{t_k} \frac{(t_k - \tau)^2}{2} \, w(\tau) \, d\tau \\[2ex] \int_{t_{k-1}}^{t_k} e^{-\beta(t_k - \tau)} \sqrt{2\sigma^2 \beta} \, v(\tau) \, d\tau \\[2ex] \int_{t_{k-1}}^{t_k} \frac{[1 - e^{-\beta(t_k - \tau)}]}{\beta} \sqrt{2\sigma^2 \beta} \, v(\tau) \, d\tau \\[2ex] \int_{t_{k-1}}^{t_k} \frac{[\beta(t_k - \tau) - 1 + e^{-\beta(t_k - \tau)}]}{\beta^2} \sqrt{2\sigma^2 \beta} \, v(\tau) \, d\tau \end{bmatrix} \qquad (6.51)$$

$$\underline{z}_k = L_k \qquad (6.52)$$

$$M_k = [\, 0 \quad \frac{4}{\Delta t} \quad \frac{-6}{\Delta t^2} \quad 0 \quad \frac{4}{\Delta t} \quad \frac{-6}{\Delta t^2} \,] \qquad (6.53)$$

$$N_k = [\, 0 \quad \frac{2}{\Delta t} \quad \frac{6}{\Delta t^2} \quad 0 \quad \frac{2}{\Delta t} \quad \frac{6}{\Delta t^2} \,] \qquad (6.54)$$

$$\underline{v}_k = 0 \qquad (6.55)$$

Finally the measurement noise covariance matrix can be determined as

$$V_k \overset{\Delta}{=} E\{\underline{v}_k \, \underline{v}_k^T\}$$

$$= 0 \qquad (6.56)$$

The plant noise covariance matrix can be determined from

$$H_{k-1} \overset{\Delta}{=} E\{\underline{w}_{k-1} \ \underline{w}_{k-1}^T\} \tag{6.57}$$

and Equation 6.51 where equations

$$E\{w(\tau_1) \ w(\tau_2)\} = \alpha\delta(\tau_1 - \tau_2) \tag{6.4}$$

$$E\{v(\tau_1) \ v(\tau_2)\} = \delta(\tau_1 - \tau_2) \tag{6.8}$$

give the statistics of the additive white inputs. Since H_{k-1} is a 6 x 6 dimension matrix, let it be partitioned to form

$$H_{k-1} = \begin{bmatrix} G_{k-1} & \vdots & 0 \\ - - - - & \vdots & - - - - \\ 0 & \vdots & J_{k-1} \end{bmatrix} \tag{6.58}$$

for convenient notation where G_{k-1} and J_{k-1} are 3 x 3 dimension matrices. The zero elements follow directly from the fact that $w(t)$ and $v(t)$ are uncorrelated. The matrix G_{k-1} is given by

$$G_{k-1}(1,1) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} E\{w(\tau_1) \ w(\tau_2)\} d\tau_2 \ d\tau_1$$

$$= \alpha\Delta t$$

$$G_{k-1}(2,2) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k-\tau_1)(t_k-\tau_2) \ E\{w(\tau_1)w(\tau_2)\} d\tau_2 \ d\tau_1$$

$$= \alpha \ \frac{\Delta t^3}{3}$$

$$G_{k-1}(3,3) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{(t_k-\tau_1)^2}{2} \frac{(t_k-\tau_2)^2}{2} E\{w(\tau_1)w(\tau_2)\} d\tau_2 \ d\tau_1$$

$$= \alpha\frac{\Delta t^5}{20}$$

$$G_{k-1}(1,2) = G_{k-1}(2,1)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_2) E\{w(\tau_1) w(\tau_2)\} \, d\tau_2 \, d\tau_1$$

$$= \alpha \frac{\Delta t^2}{2}$$

$$G_{k-1}(1,3) = G_{k-1}(3,1)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{(t_k - \tau_2)^2}{2} E\{w(\tau_1) w(\tau_2)\} \, d\tau_2 \, d\tau_1$$

$$= \alpha \frac{\Delta t^3}{6}$$

$$G_{k-1}(2,3) = G_{k-1}(3,2)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (t_k - \tau_1) \frac{(t_k - \tau_2)^2}{2} E\{w(\tau_1)w(\tau_2)\} d\tau_2 \, d\tau_1$$

$$= \alpha \frac{\Delta t^4}{8}$$

Therefore

$$G_{k-1} = \begin{bmatrix} \alpha \Delta t & \alpha \dfrac{\Delta t^2}{2} & \alpha \dfrac{\Delta t^3}{6} \\[2ex] \alpha \dfrac{\Delta t^2}{2} & \alpha \dfrac{\Delta t^3}{3} & \alpha \dfrac{\Delta t^4}{8} \\[2ex] \alpha \dfrac{\Delta t^3}{6} & \alpha \dfrac{\Delta t^4}{8} & \alpha \dfrac{\Delta t^5}{20} \end{bmatrix} \qquad (6.59)$$

$$J_{k-1}(1,1) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} 2\sigma^2\beta \; e^{-\beta(t_k-\tau_1)} e^{-\beta(t_k-\tau_2)}$$

$$E\{v(\tau_1) \; v(\tau_2)\} \; d\tau_2 \; d\tau_1$$

$$= \sigma^2(1 - e^{-2\beta\Delta t}) \tag{6.60}$$

$$J_{k-1}(2,2) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{2\sigma^2}{\beta} [1 - e^{-\beta(t_k-\tau_1)}][1 - e^{-\beta(t_k-\tau_2)}]$$

$$E\{v(\tau_1) \; v(\tau_2)\} \; d\tau_2 \; d\tau_1$$

$$= \frac{2\sigma^2}{\beta} [\Delta t + \frac{1}{2\beta} - \frac{1}{2\beta} (2 - e^{-\beta\Delta t})^2] \tag{6.61}$$

$$J_{k-1}(3,3) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{2\sigma^2}{\beta^3}[\beta(t_k-\tau_1) - 1 + e^{-\beta(t_k-\tau_1)}]$$

$$[\beta(t_k-\tau_2) - 1 + e^{-\beta(t_k-\tau_2)}]E\{v(\tau_1) \; v(\tau_2)\} \; d\tau_2 \; d\tau_1$$

$$= \frac{2\sigma^2}{\beta^3} [\frac{1}{2\beta} + \frac{\beta^2\Delta t^3}{3} - \beta\Delta t^2 + \Delta t - 2\Delta t \; e^{-\beta\Delta t} - \frac{e^{-2\beta\Delta t}}{2\beta}] \tag{6.62}$$

$$J_{k-1}(1,2) = J_{k-1}(2,1)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} 2\sigma^2 \; e^{-\beta(t_k-\tau_1)}[1 - e^{-\beta(t_k-\tau_2)}]$$

$$E\{v(\tau_1) \; v(\tau_2)\} \; d\tau_2 \; d\tau_1$$

$$= \frac{\sigma^2}{\beta} (1 - e^{-\beta\Delta t})^2 \tag{6.63}$$

$$J_{k-1}(1,3) = J_{k-1}(3,1)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{2\sigma^2}{\beta} e^{-\beta(t_k-\tau_1)} [\beta(t_k-\tau_2) - 1$$

$$+ e^{-\beta(t_k-\tau_2)}] \, E\{v(\tau_1) \, v(\tau_2)\} \, d\tau_2 \, d\tau_1$$

$$= \frac{2\sigma^2}{\beta} \left[\frac{1}{2\beta} - \Delta t \, e^{-\beta\Delta t} - \frac{e^{-2\beta\Delta t}}{2\beta} \right] \tag{6.64}$$

$$J_{k-1}(2,3) = J_{k-1}(3,2)$$

$$= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \frac{2\sigma^2}{\beta^2} [1 - e^{-\beta(t_k-\tau_1)}][\beta(t_k-\tau_2) - 1$$

$$+ e^{-\beta(t_k-\tau_2)}] \, E\{v(\tau_1) \, v(\tau_2)\} \, d\tau_2 \, d\tau_1$$

$$= \frac{2\sigma^2}{\beta^2} \left[\frac{1}{2\beta} + \frac{\beta\Delta t^2}{2} - \Delta t - \frac{e^{-\beta\Delta t}}{\beta} + \Delta t \, e^{-\beta\Delta t} + \frac{e^{-2\beta\Delta t}}{2\beta} \right] \tag{6.65}$$

This completes the evaluation of all necessary quantities for the implementation of the delayed-state Kalman filtering Equations 3.21, 3.26, 3.27, and 3.28.

D. Continuous Kalman-Bucy Filter

Kalman-Bucy filter equations exist for this example with Markov measurement noise since it is a continuous system as shown in Figure 10. Methods are outlined in Appendix B for determining the optimal gain matrix and the continuous error

covariance matrix for the optimal gain case when the measurements are corrupted with colored noise. As in the white measurement noise example, the steady-state value of the continuous error-covariance for state u(t) in this example represents a lower bound which is approached only from above by discrete filters as Δt approaches zero.

Formulating the augmented system Equations 6.10 and 6.11 of the example in terms of Equations 11.6 and 11.7 results in

$$\begin{bmatrix} \underline{x}(t) \\ \underline{n}(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ n(t) \end{bmatrix} \tag{6.66}$$

$$F(t) = 0 \tag{6.67}$$

$$A(t) = -\beta \tag{6.68}$$

$$\underline{w}(t) = w(t) \tag{6.69}$$

$$\underline{v}(t) = \sqrt{2\sigma^2\beta} \; v(t) \tag{6.70}$$

$$\underline{z}(t) = z(t) \tag{6.71}$$

$$\underline{M}(t) = 1 \tag{6.72}$$

From Equation 11.4 if

$$E\left\{ \begin{bmatrix} \underline{w}(t) \\ \underline{v}(t) \end{bmatrix} \begin{bmatrix} \underline{w}(\tau) \\ \underline{v}(\tau) \end{bmatrix}^T \right\} = E\left\{ \begin{bmatrix} w(t) \\ \sqrt{2\sigma^2\beta} \; v(t) \end{bmatrix} [w(\tau) \; \sqrt{2\sigma^2\beta} \; v(\tau)] \right\}$$

$$= \begin{bmatrix} \alpha\delta(t-\tau) & 0 \\ 0 & 2\sigma^2\beta \; \delta(t-\tau) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & 0 \\ 0 & 2\sigma^2\beta \end{bmatrix} \delta(t-\tau) \tag{6.73}$$

then

$$Q(t) = \alpha \qquad (6.74)$$

$$R(t) = 2\sigma^2\beta \qquad (6.75)$$

Equations 11.24 and 11.25 become

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad (6.76)$$

and

$$T^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad (6.77)$$

Thus

$$\begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} = T \begin{bmatrix} u(t) \\ n(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ n(t) \end{bmatrix} \qquad (6.78)$$

or $z(t) \equiv u(t) + n(t)$ and $\xi(t) \equiv u(t)$ implies that the state $u(t)$ is being estimated directly. Of course the definition for $J(t)$, Equation 11.33, must also apply for the associated measurement as

$$J(t) = M(t) \ F(t) - A(t) \ M(t) + \dot{M}(t)$$

$$= (1) \ (0) - (-\beta) \ (1) + (0)$$

$$= \beta \qquad (6.79)$$

The optimal gain matrix $K(t)$ for this example is given by the above results and Equation 11.44 as

$$K(t) = [P(t) \ J^T(t) + Q(t) \ M^T(t)][M(t) \ Q(t) \ M^T(t) + R(t)]^{-1}$$

$$= (\beta P(t) + \alpha) \ (\alpha + 2\sigma^2\beta)^{-1} \qquad (6.80)$$

For the optimal gain case the estimation error-covariance matrix is given as the solution of the matrix differential Equation 11.45 as

$$\dot{P}(t) = F(t)\ P(t) + P(t)\ F^T(t) + Q(t) - K(t)$$

$$[M(t)\ Q(t)\ M^T(t) + R(t)]\ K^T(t)$$

$$= 0 + 0 + \alpha - \frac{(\beta P(t) + \alpha)}{(\alpha + 2\sigma^2\beta)}\ (\alpha + 2\sigma^2\beta)\ \frac{(\beta P(t) + \alpha)}{(\alpha + 2\sigma^2\beta)}$$

$$= \alpha - \frac{(\beta P(t) + \alpha)^2}{(\alpha + 2\sigma^2\beta)}$$

$$= \alpha - \frac{(\beta P(t) + \alpha)^2}{(\alpha + \gamma/\alpha)}$$

$$= a - b\ P^2(t) - c\ P(t) \tag{6.81}$$

where

$$\gamma = 2\alpha\sigma^2\beta \tag{6.82}$$

$$a = \frac{\gamma}{\alpha + (\gamma/\alpha)} \tag{6.83}$$

$$b = \frac{\beta^2}{\alpha + (\gamma/\alpha)} \tag{6.84}$$

$$c = \frac{2\alpha\ \beta}{\alpha + (\gamma/\alpha)} \tag{6.85}$$

Equation 6.81 must be solved in order that P_∞ can be determined; therefore, assume a solution of the form

$$P(t) = \frac{f + g\ e^{-t}}{h + i\ e^{-t}} \tag{6.86}$$

where f, g, h, and i are arbitrary constants. By substituting

this equation into Equation 6.81 and equating like terms, Equation 6.86 is found to be a valid solution when

$$h = \frac{(c + \sqrt{c^2 + 4\,a\,b})\,f}{2a} \tag{6.87}$$

$$i = \frac{(c + \sqrt{c^2 + 4\,a\,b})\,g}{2a} \tag{6.88}$$

The assumed form of $P(t)$ allows P_∞ to be easily evaluated from Equations 6.82 through 6.87 as

$$
\begin{aligned}
P_\infty &= \underset{t \to \infty}{\text{limit}}\ P(t) \\[2mm]
&= \frac{f}{h} \\[2mm]
&= \frac{2a}{c + \sqrt{c^2 + 4\,a\,b}} \\[2mm]
&= \frac{\dfrac{2\gamma}{(\alpha + \frac{\gamma}{\alpha})}}{\dfrac{2\alpha\beta}{(\alpha + \frac{\gamma}{\alpha})} + \sqrt{\dfrac{4\alpha^2\beta^2}{(\alpha + \frac{\gamma}{\alpha})^2} + \dfrac{4\gamma\beta^2}{(\alpha + \frac{\gamma}{\alpha})^2}}} \\[2mm]
&= \frac{\gamma}{\beta(\sqrt{\alpha^2 + \gamma} + \alpha)} \cdot \frac{\sqrt{\alpha^2 + \gamma} - \alpha}{\sqrt{\alpha^2 + \gamma} - \alpha} \\[2mm]
&= \frac{\sqrt{\alpha^2 + \gamma} - \alpha}{\beta} \tag{6.89}
\end{aligned}
$$

where

$$\gamma = 2\alpha\sigma^2\beta \tag{6.82}$$

The steady-state value of the error-covariance for the state $u(t)$ in this example with white plant noise and Markov

measurement noise is strictly a function of these noise parameters.

E. Results

Computation of the discrete Kalman, interval-averaging, and linearized-sampling filters as evaluated in the preceding Sections A, B, and C for the Markov measurement noise example was performed by digital computer. The determination of the steady-state value P_∞ of the continuous error-covariance for state $u(t)$ in this example is merely a matter of solving Equations 6.82 and 6.89 for the various noise parameters α, σ^2, and β. A computer solution was not required for this value.

As was the case for the white measurement noise example, the three discrete filters requiring computer solutions were programmed using Fortran IV language and processed on the Iowa State University IBM-360 Model 65 computer. After choosing α, σ^2, β, Δt, $k = 1$ and $P_0 = 0$, iterations on k were performed until the a posteriori estimation error-covariance, $P_k(1,1)$, for the state $u(t)$ reached a steady state value defined as $P_{ss}(1,1)$. The error-covariance element of the state $u(t)$ was in all cases the $(1,1)$ element. Again the quantity Q_k in Equation 3.26 was prevented from going to zero by performing all computations in double precision. The value Q_k must be nonsingular since its inverse is required in Equation 3.27. The initial error-covariance matrix was set equal to zero by the equation $P_0 = 0$. The noise parameters were selected to

represent the relative effects of various noise amplitudes and correlation times on $P_{ss}(1,1)$ as the discrete time interval varied.

The computed steady-state a posteriori estimation error-covariance value $P_{ss}(1,1)$ of state $u(t)$ for the three discrete filters is shown plotted versus the discrete time interval in Figures 12 to 18 where the selected values of α, σ^2 and β are as indicated on each figure. The lower bound of the steady-state continuous error-covariance value, P_∞, is also noted.

As the time interval approaches zero observe that for each set of noise parameters considered the value $P_{ss}(1,1)$ for all three discrete filters approaches P_∞ from the Kalman-Bucy filter. The linearized-sampling filter value of $P_{ss}(1,1)$ drifts away from P_∞ similar to the white measurement noise example for $0.01 < \Delta t < 0.1$ where the linear least square fit in this process approximates the noise rather than the trend of the signal, but then it returns towards P_∞ at smaller values of Δt. The larger correlation times, i.e., smaller β, appear to reduce thus drift away from P_∞. For larger β, smaller correlation time, $P_{ss}(1,1)$ for both prefiltering processes remains close to the lower limit P_∞ for the range of $0 < \Delta t < 1$.

Probably one of the most important results of this work is the fact none of the three discrete filters is "best" over
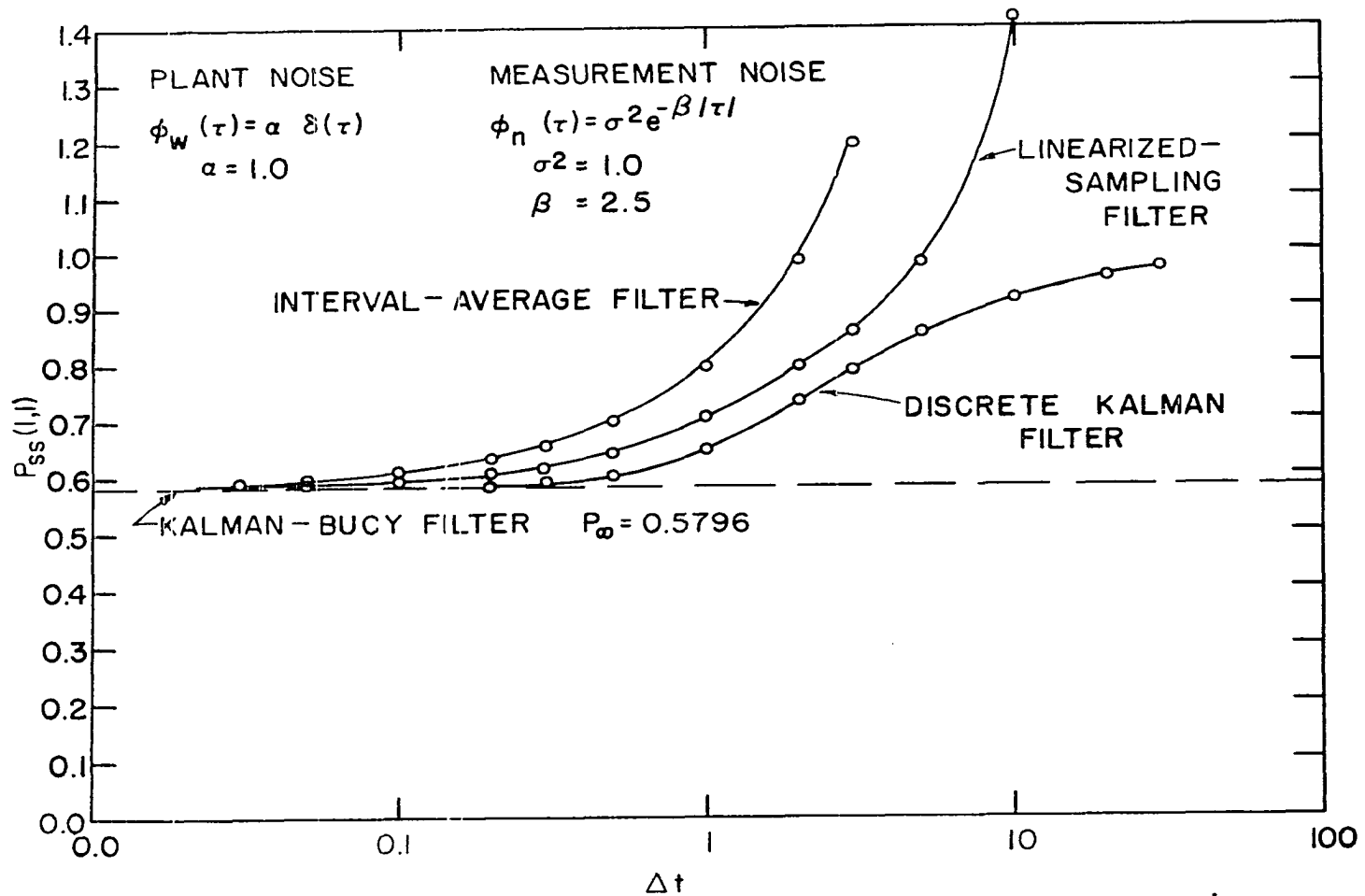
Figure 12. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for Markov measurement noise example of Figure 10
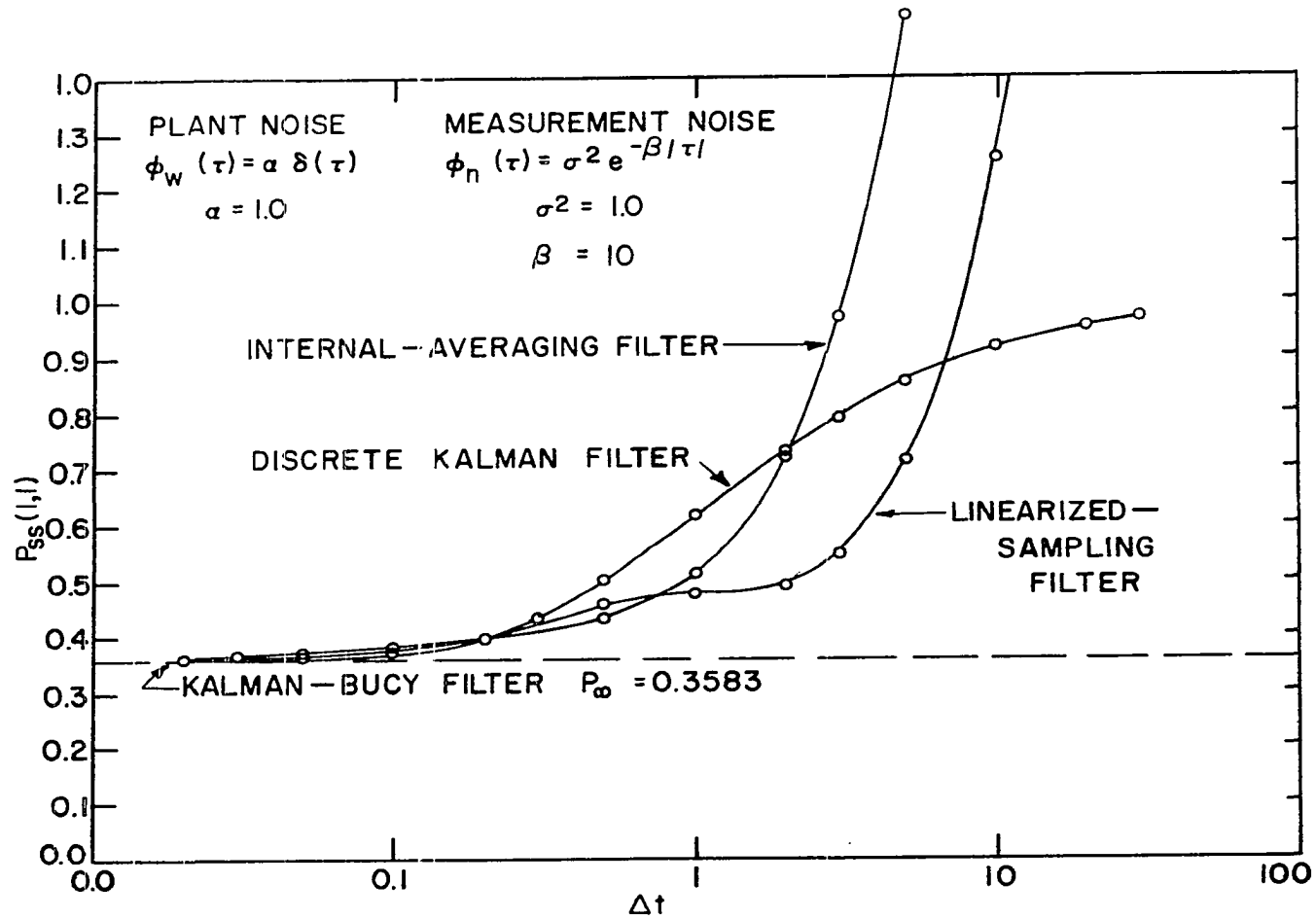
Figure 13. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for Markov measurement noise example of Figure 10

PLANT NOISE

$\phi_W(\tau) = \alpha \, \delta(\tau)$

$\alpha = 1.0$

MEASUREMENT NOISE

$\phi_n(\tau) = \sigma^2 \, e^{-\beta|\tau|}$

$\sigma^2 = 1.0$

$\beta = 100$

INTERVAL—AVERAGING FILTER

LINEARIZED—SAMPLING FILTER

DISCRETE KALMAN FILTER

KALMAN—BUCY FILTER $P_\infty = 0.1318$

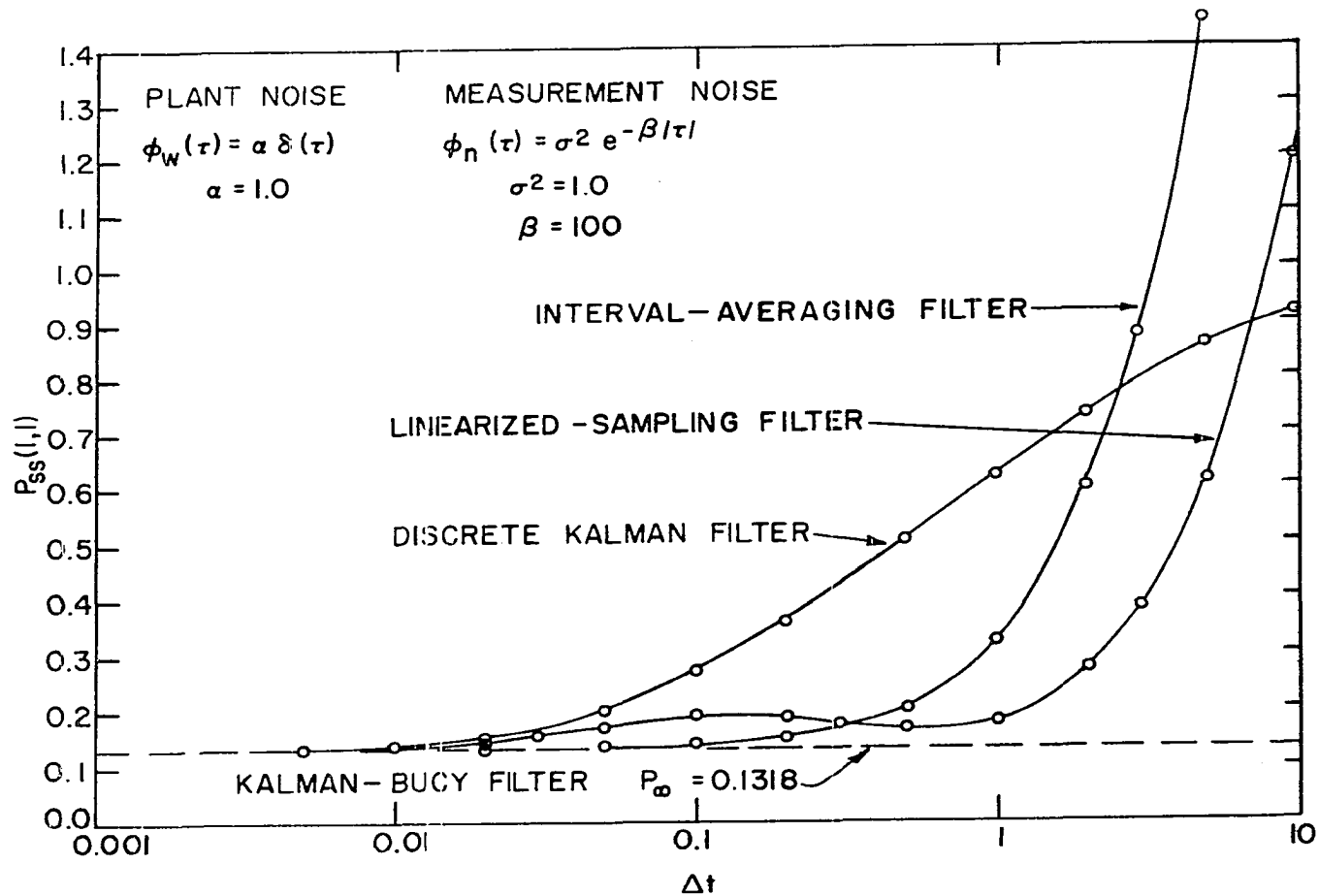$P_{ss}(1,1)$

Δt

Figure 14.   Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for Markov measurement noise example of Figure 10

Figure 15. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state u(t) for Markov measurement noise example of Figure 10
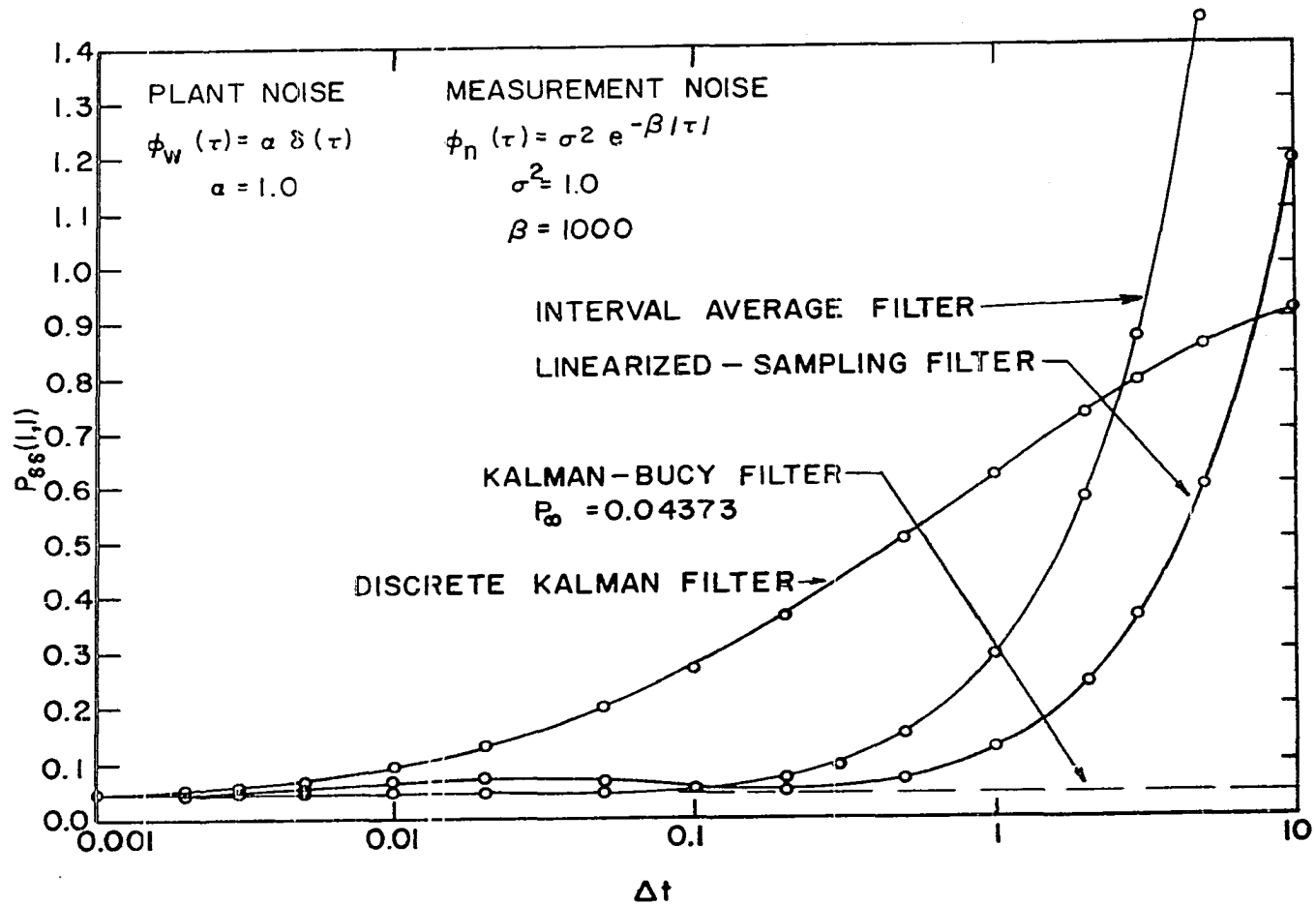
Figure 16. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for Markov measurement noise example of Figure 10

Figure 17. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for Markov measurement noise example of Figure 10
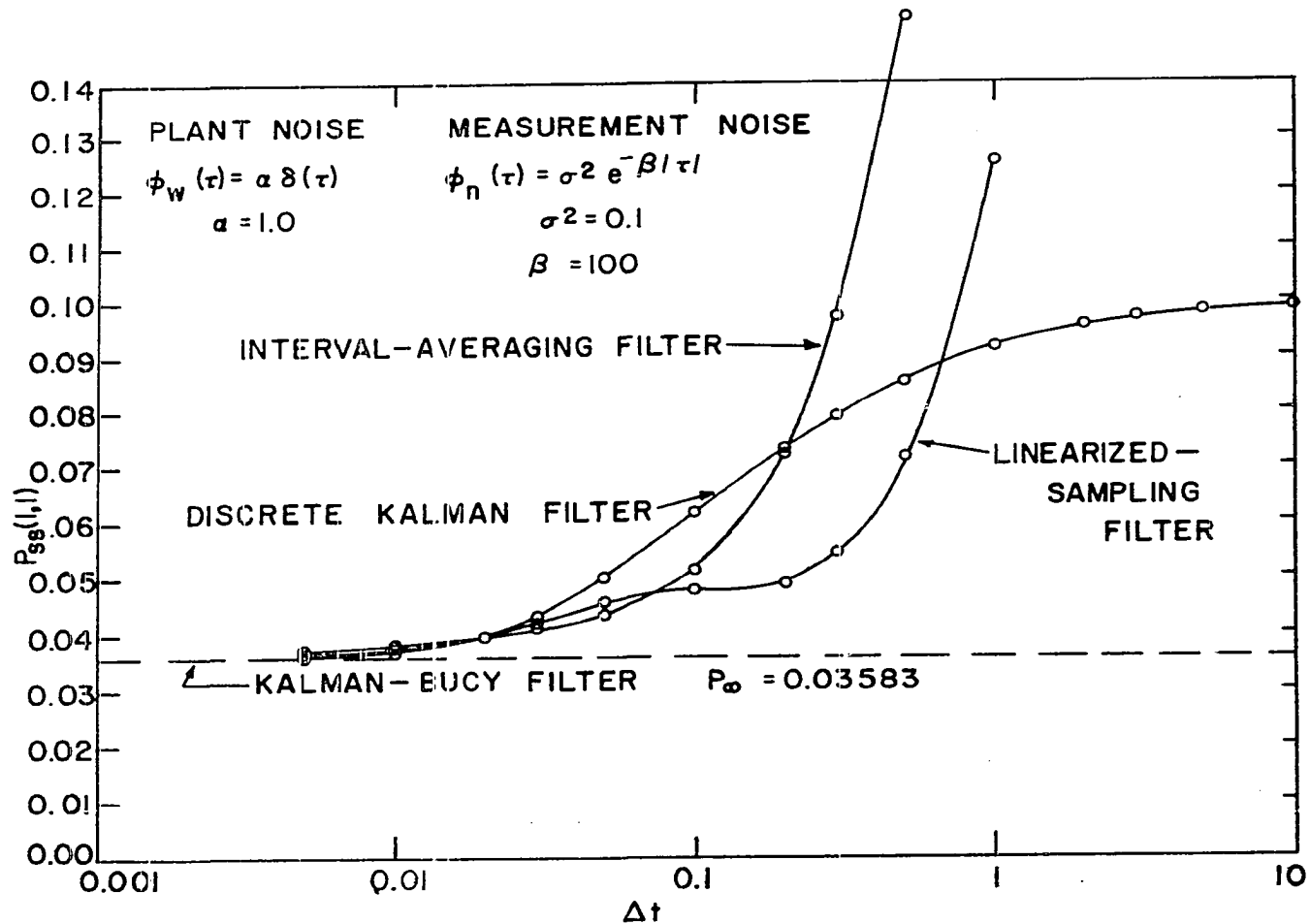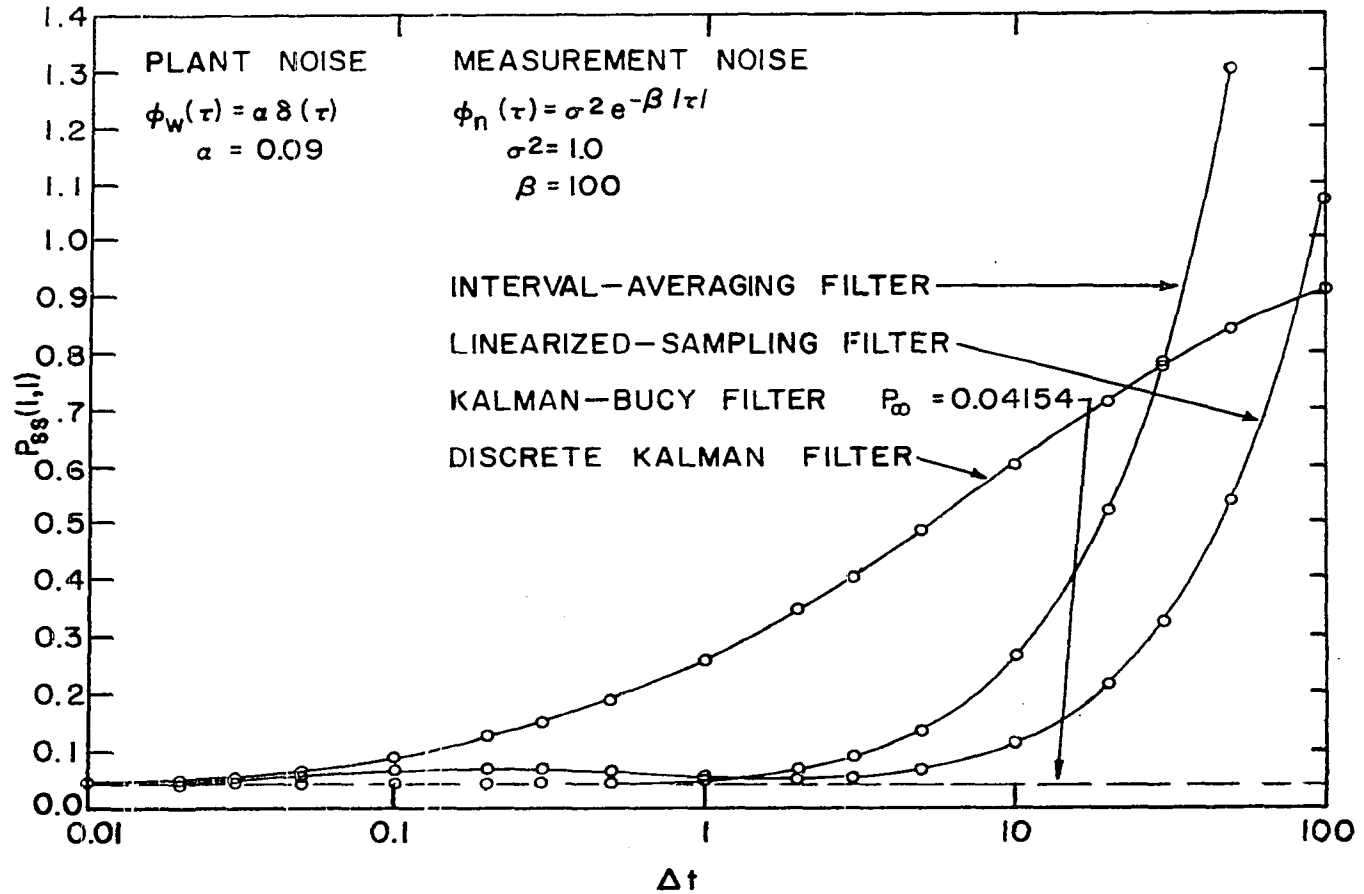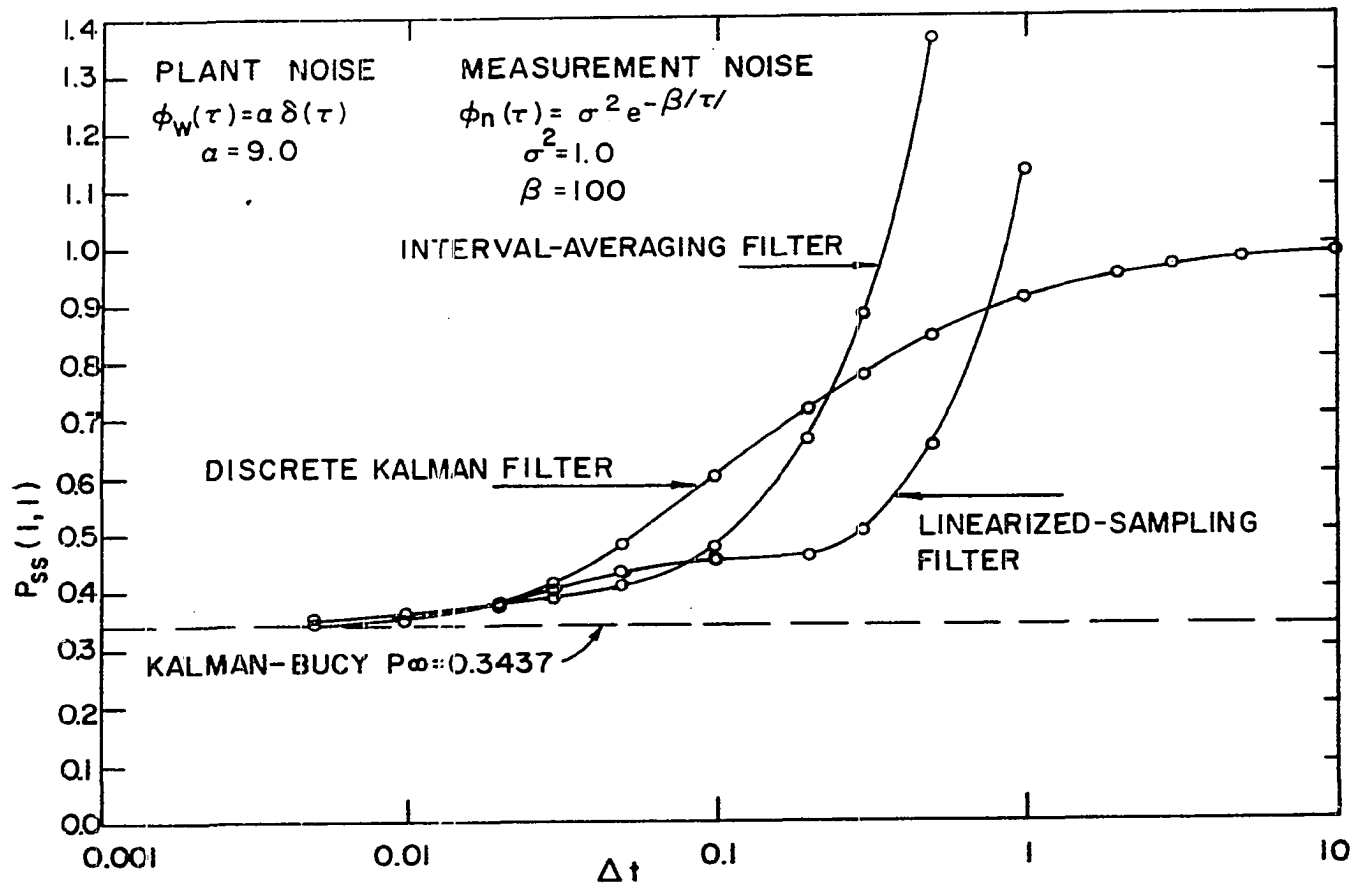
Figure 18. Computed steady-state a posteriori estimation error-covariance $P_{ss}(1,1)$ of state $u(t)$ for Markov measurement noise example of Figure 10

the whole range of discrete time intervals. The interval-averaging filter produces the lowest value of steady-state covariance error over an approximate discrete time interval range of $0 < \Delta t < 1$; in fact, for large values of β this process offers a considerable improvement over the discrete Kalman filter. The linearized-sampling process yields the best filter for the discrete time range of approximately $1 < \Delta t < 10$, and it too shows a sizeable reduction in $P_{ss}(1,1)$ from the values obtained by either of the other two filters. Only when the discrete time interval exceeds a value of approximately 10 does the discrete Kalman filter offer the lowest $P_{ss}(1,1)$ value. In other words, the correct or best discrete filter to use in analyzing a system similar to this example, where continuous measurement data is corrupted by Markov noise, depends primarily on the discrete time interval that is selected.

Only for the case of very large correlation time or smaller β, Figure 12, is there no improvement of the interval-averaging and linearized-sampling filters over the discrete Kalman filter.

VII. SUMMARY AND CONCLUSIONS

The derivations in Chapter IV developed two methods of processing continuous noisy measurement data in a discrete Kalman filter. The first method simply averages the continuous measurements over the discrete time interval. In the second method a linear least square approximation of the data over the interval is sampled at the end of the interval to determine an equivalent noisy measurement. Examples with white and Markov measurement noise were evaluated using these new techniques as well as by the usual Kalman and Kalman-Bucy techniques for comparison. The results of these evaluations shown in Chapters V and VI demonstrate how the noise amplitudes and noise correlation times affect the steady-state a posteriori estimation covariance-error value as the discrete time interval varies.

A noteworthy contribution of this work is the method of analysis using the delayed state model. The interval-averaging and linearized-sampling prefilters process only the continuous measurements to reduce unwanted measurement noise. By a judicious selection of state variables, the prefilter is incorporated into the continuous system yielding a modified discrete plant and measurement model which is equivalent to the model of the Kalman filter with delayed states as observables. The delayed-state model allows the interval-averaging and linearized-sampling filters to be treated by conventional

recursive techniques using a digital computer.

The results of only the Markov measurement noise example are mentioned here since this type of system is more realistic from a practical standpoint. Neither prefiltering process improves the discrete Kalman filter when the correlation time of the measurement noise approaches the discrete time interval. But as the correlation time is reduced relative to the time interval a significant improvement is noted in both methods. And it is seen that the effective measurement noise is substantially reduced over particular ranges of the discrete time interval. Each range is somewhat dependent upon the specific noise parameter values but in general terms the interval-averaging filter should be used for the smallest time intervals less than one. The linearized-sampling filter extends the effective measurement noise reduction from time intervals near one to ten and larger. For time intervals greater than this, the discrete Kalman filter should be used. In other words, none of the three discrete filters is "best" over the whole range of discrete time intervals. Each discrete filter is applicable over a specific range of the discrete time interval.

Thus, prefiltering or preprocessing of continuous noisy measurement data by the interval-averaging or linearized-sampling techniques can improve the results of discrete Kalman filtering by reducing the effective measurement noise. Com-

putation time may even be reduced if the discrete time interval can be greater using the prefiltering techniques to produce results equivalent to those of the discrete Kalman filter. For some cases when the discrete time interval is not extremely large the results of the preprocessing discrete filters are comparable to those of the continuous Kalman-Bucy filter.

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X. APPENDIX A

A heuristic derivation of continuous Kalman-Bucy filter equations and the general solution of the error-covariance matrix differential equation are outlined below for a system with white plant noise and white measurement noise (6).

As stated previously in Chapter III, the solution provides the unbiased, minimum variance estimate of the state $\underline{x}(t)$ from measurement data $\underline{z}(t)$ based on the linear system given by

$$\dot{\underline{x}}(t) = F(t)\ \underline{x}(t) + \underline{w}(t) \tag{10.1}$$

$$\underline{z}(t) = M(t)\ \underline{x}(t) + \underline{v}(t) \tag{10.2}$$

where

$$E\{\underline{w}(t)\ \underline{w}^T(\tau)\} = Q(t)\delta(t-\tau) \text{ and } E\{\underline{v}(t)\ \underline{v}^T(\tau)\} = R(t)\delta(t-\tau)$$

Denote an estimate of the state $\underline{x}(t_0)$ known at t_0 and based upon measurement data $\underline{z}(t_0)$ as $\hat{\underline{x}}(t_0|t_0)$. Referring to Equation 10.1 and noting the white noise forcing function $\underline{w}(t)$, the estimate is described by

$$\dot{\hat{\underline{x}}}(t|t_0) = F(t)\ \hat{\underline{x}}(t|t_0) \qquad t > t_0 \tag{10.3}$$

in the absence of additional data. The availability of measurement data $\underline{z}(t)$ after t_0 and the determination of the expected measurement from Equation 10.2 as

$$\hat{\underline{z}}(t|t) = M(t)\ \hat{\underline{x}}(t|t) \tag{10.4}$$

implies that there exists a "residual" difference between them.

$$\underline{r}(t) = \underline{z}(t) - \hat{\underline{z}}(t|t) \qquad (10.5)$$

This contribution is considered to indicate the error in the estimate $\hat{\underline{x}}(t|t)$ and as such is added to Equation 10.3 after proper weighting by an unknown gain matrix K(t). Thus the unbiased minimum variance estimate of the linear system described by Equations 10.1 and 10.2 is given by the solution of the system

$$\dot{\hat{\underline{x}}}(t|t) = F(t)\hat{\underline{x}}(t|t) + K(t)[\underline{z}(t) - M(t)\hat{\underline{x}}(t|t)] \text{ for } t > t_0 \qquad (10.6)$$

where $\hat{\underline{x}}(t_0|t_0)$ is known. It should be observed that the estimate given by Equation 10.6 is unbiased meaning that

$$E\{\underline{x}(t)\} = E\{\hat{\underline{x}}(t|t)\}$$

is true if the initial condition $\hat{\underline{x}}(t_0|t_0)$ satisfies the requirement

$$E\{\underline{x}(t_0)\} = E\{\hat{\underline{x}}(t_0|t_0)\} \qquad (10.7)$$

As was the case with the discrete Kalman filter derivations, the gain matrix K(t) in the Kalman-Bucy filter, Equation 10.6 will be chosen to minimize the loss function. Make the following definitions:

$$\underline{e}(t|t) \overset{\Delta}{=} \hat{\underline{x}}(t|t) - \underline{x}(t) = \text{error of estimate}$$

$$P(t|t) \overset{\Delta}{=} E\{\underline{e}(t|t)\underline{e}^T(t|t)\} = \text{error covariance matrix}$$

$$L \overset{\Delta}{=} \text{trace } P(t|t) = \text{loss function}$$

Completing the details as described above using these definitions leads to the remaining equations for P(t|t) and K(t).

The Kalman-Bucy filter equations include Equations 10.6 and 10.7 as well as the optimal gain matrix given by

$$K(t) = P(t \mid t)M^T(t)R^{-1}(t) \tag{10.8}$$

and the error covariance matrix for the optimal gain case is given as the solution of the matrix differential equation

$$\dot{P}(t \mid t) = P(t \mid t)F^T(t) + F(t) P(t \mid t)$$

$$- P(t \mid t)M^T(t)R^{-1}(t)M(t)P(t \mid t) + Q(t) \tag{10.9}$$

with known initial condition $P(t_0) = P_0$.

A matrix differential equation of the general type given by Equation 10.9 is called a matrix Ricatti equation. The general solution to this equation has been determined and is discussed below.

The general matrix-Ricatti equation has the form

$$\dot{W}(t) = W(t)A^T(t) + A(t)W(t) + W(t)B(t)W(t) + C(t) \tag{10.10}$$

where $W(t_0) = W_0$ is a non-negative definite matrix. Also $A(t)$, $B(t)$ and $C(t)$ are nxn matrices of continuous functions with $B(t)$ and $C(t)$ being non-negative definite for $t > t_0$. Using the method of Sorenson and Stubberud (6) consider the set of equations

$$\dot{Y}(t) = A(t)Y(t) + C(t)Z(t); \quad Y(t_0) = W_0 \tag{10.11}$$

$$\dot{Z}(t) = -B(t)Y(t) - A^T(t)Z(t); \quad Z(t_0) = I \tag{10.12}$$

where

$$Y(t) = W(t)Z(t) \tag{10.13}$$

and therefore

$$\dot{Y}(t) = \dot{W}(t)Z(t) + W(t)\dot{Z}(t) \tag{10.14}$$

Combining this equation with Equations 10.11 and 10.12 yields

$$A(t)Y(t) + C(t)Z(t) = \dot{W}(t)Z(t) - W(t)B(t)Y(t) - W(t)A^T(t)Z(t)$$

$$(10.15)$$

When the substitution of $W(t)Z(t)$ for $Y(t)$ is made, the result is

$$[\dot{W}(t) - W(t)A^T(t) - A(t)W(t) - W(t)B(t)W(t) - C(t)]Z(t) = 0$$

$$(10.16)$$

Assuming $Z(t)$ is nonsingular for all $t > t_0$ implies that

$$\dot{W}(t) = W(t)A^T(t) + A(t)W(t) + W(t)B(t)W(t) + C(t) \qquad (10.17)$$

but Equations 10.10 and 10.17 are identical. It follows then that Equation 10.13 is satisfied by $Y(t)$ and $Z(t)$ and, if $Z(t)$ is nonsingular for all $t > t_0$, that

$$W(t) = Y(t)Z^{-1}(t) \qquad (10.18)$$

is the general solution to Equation 10.10 with

$$W(t_0) = Y(t_0)Z^{-1}(t_0) = (W_0)(I) = W_0 \qquad (10.19)$$

The matrix $Z(t)$ is a transition matrix describing the dynamics and satisfying

$$\dot{Z}(t) = -A^T(t)Z(t) - B(t)Y(t)$$

$$\frac{dZ(t)}{dt} = [-A^T(t) - B(t)W(t)]Z(t) \qquad (10.20)$$

and

$$Z(t_0) = I \qquad (10.21)$$

therefore $Z(t)$ has an inverse and is nonsingular as does a transition matrix. This justifies the assumption made in developing Equation 10.17.

Referring back to the Kalman-Bucy filter, Equations 10.8 and 10.9, and comparing them to Equation 10.10 note that

$$W(t) = P(t \mid t)$$

$$A(t) = F(t)$$

$$B(t) = -M^T(t)R^{-1}(t)M(t)$$

$$C(t) = Q(t)$$

$$W(t_0) = P(t_0) = P_0$$

Thus the equivalent set of equations for Equations 10.11 and 10.12 becomes

$$\dot{Y}(t) = F(t)Y(t) + Q(t)Z(t); \quad Y(t_0) = P_0 \qquad (10.22)$$

$$\dot{Z}(t) = M^T(t)R^{-1}(t)M(t)Y(t) - F^T(t)Z(t); \quad Z(t_0) = I \quad (10.23)$$

and combining them into one matrix equation shows that

$$\begin{bmatrix} \dot{Y}(t) \\ \dot{Z}(t) \end{bmatrix} = \begin{bmatrix} F(t) & Q(t) \\ M^T(t)R^{-1}(t)M(t) & -F^T(t) \end{bmatrix} \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} \qquad (10.24)$$

Define the transition matrix

$$\Phi(t,t_0) \overset{\Delta}{=} \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix} \qquad (10.25)$$

where $\Phi(t,t_0)$ is the solution of the matrix differential equation

$$\frac{d\Phi(t,t_0)}{dt} = \begin{bmatrix} F(t) & Q(t) \\ M^T(t)R^{-1}(t)M(t) & -F^T(t) \end{bmatrix} \Phi(t,t_0) \qquad (10.26)$$

Therefore the solution to Equation 10.24 is

$$\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} = \Phi(t,t_0) \begin{bmatrix} Y(t_0) \\ Z(t_0) \end{bmatrix}$$

$$= \Phi(t,t_0) \begin{bmatrix} P_0 \\ I \end{bmatrix}$$

$$= \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix} \begin{bmatrix} P_0 \\ I \end{bmatrix} \tag{10.27}$$

or

$$Y(t) = \phi_{11}(t,t_0) P_0 + \phi_{12}(t,t_0) \tag{10.28}$$

$$Z(t) = \phi_{21}(t,t_0) P_0 + \phi_{22}(t,t_0) \tag{10.29}$$

Finally from Equations 10.18, 10.28, and 10.29

$$P(t \mid t) = Y(t) Z^{-1}(t)$$

$$= [\phi_{11}(t,t_0) P_0 + \phi_{12}(t,t_0)] \cdot$$

$$[\phi_{21}(t,t_0) P_0 + \phi_{22}(t,t_0)]^{-1} \tag{10.30}$$

XI. APPENDIX B

A special case of the continuous Kalman-Bucy filter is considered here for a system with white plant noise as in Appendix A but with colored measurement noise (6). The problem is reformulated by state vector augmentation to form a system in which only white-noise appears explicitly. The derivation retains the original unaugmented state vector $\underline{x}$ so that estimates are made directly on $\underline{x}$.

Recall that special treatment of the colored measurement noise problem is required because of the components of the measurement vector which contain only colored-noise. Unfortunately, after the use of shaping filters and state vector augmentation, the unaugmented colored measurement noise elements become zero elements in the augmented measurement noise vector. This prevents the covariance matrix, $R(t)$, of the augmented measurement vector from being positive-definite. Thus $R^{-1}(t)$ does not exist and the optimal gain matrix cannot be evaluated. The usual Kalman-Bucy equations therefore cannot be used when the measurement noise is colored.

Let a system be given as

$$\underline{\dot{x}}(t) = F(t) \ \underline{x}(t) + \underline{w}(t) \tag{11.1}$$

$$\underline{z}(t) = M(t) \ \underline{x}(t) + \underline{n}(t) \tag{11.2}$$

where $\underline{n}(t)$ is a process of zero mean colored-noise given by the shaping filter

$$\underline{\dot{n}}(t) = A(t)\underline{n}(t) + \underline{v}(t) \tag{11.3}$$

where $A(t)$ and the statistics of the white noise $\underline{v}(t)$ are chosen such that $\underline{n}(t)$ has the prescribed statistics. The white noise processes $\underline{w}(t)$ and $\underline{v}(t)$ are uncorrelated and defined so that

$$E \left\{ \begin{bmatrix} \underline{w}(t) \\ \underline{v}(t) \end{bmatrix} \begin{bmatrix} \underline{w}(\tau) \\ \underline{v}(\tau) \end{bmatrix}^T \right\} = \begin{bmatrix} Q(t) & 0 \\ 0 & R(t) \end{bmatrix} \delta(t-\tau) \qquad (11.4)$$

also the covariance of $\underline{n}(t)$ at t_0 is

$$E\{\underline{n}(t_0)\underline{n}^T(t_0)\} = N(t_0) \qquad (11.5)$$

The augmented system is

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\underline{n}}(t) \end{bmatrix} = \begin{bmatrix} F(t) & 0 \\ 0 & \cdot \ A(t) \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{n}(t) \end{bmatrix} + \begin{bmatrix} \underline{w}(t) \\ \underline{v}(t) \end{bmatrix} \qquad (11.6)$$

$$\underline{z}(t) = [M(t) \ I] \begin{bmatrix} \underline{x}(t) \\ \underline{n}(t) \end{bmatrix} + 0 \qquad (11.7)$$

Note that the measurement in augmented form is perfect, i.e., noiseless. The solution to this problem presented hereafter is essentially that of Stear and Stubberud (13).

One aspect of this problem is discussed first before dealing with it directly. It is possible to define

$$\dot{\underline{x}}(t) = F(t) \ \underline{x}(t) + \underline{w}(t) \qquad (11.8)$$

$$\begin{bmatrix} \underline{z}_1(t) \\ \underline{z}_2(t) \end{bmatrix} = \begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix} \underline{x}(t) + \begin{bmatrix} \underline{v}(t) \\ 0 \end{bmatrix} \qquad (11.9)$$

where $\underline{w}(t)$ and $\underline{v}(t)$ are white noise and where m-vector $\underline{z}_1$ contains white-noise, p vector $\underline{z}_2$ is noise free and $\underline{x}$ is an (n+p) augmented state vector.

Now redefine the state variables with subvectors $\underline{z}_2$ and $\underline{\xi}$ so that

$$\begin{bmatrix} \underline{z}_2(t) \\ \underline{\xi}(t) \end{bmatrix} \overset{\Delta}{=} \begin{bmatrix} H_2(t) \\ H_3(t) \end{bmatrix} \underline{x}(t) = T(t)\underline{x}(t) \tag{11.10}$$

and

$$T(t) \overset{\Delta}{=} \begin{bmatrix} H_2(t) \\ H_3(t) \end{bmatrix} \tag{11.11}$$

is defined so that $T^{-1}(t)$ exists, if $H_2 H_3^T = 0$ and $H_3 H_3^T = I$. Let

$$T^{-1}(t) = [J_2(t) \; J_3(t)] \tag{11.12}$$

where

$$J_2(t) = H_2^T(t) \, (H_2(t)H_2^T(t))^{-1} \tag{11.13}$$

$$J_3(t) = H_3^T(t) \tag{11.14}$$

From Equation 11.10 and since $T^{-1}(t)$ exists, it follows that

$$\underline{x}(t) = T^{-1}(t) \begin{bmatrix} \underline{z}_2(t) \\ \underline{\xi}(t) \end{bmatrix}$$

$$= [J_2(t)J_3(t)] \begin{bmatrix} \underline{z}_2(t) \\ \underline{\xi}(t) \end{bmatrix}$$

$$= J_2(t)\underline{z}_2(t) + J_3(t)\underline{\xi}(t) \tag{11.15}$$

Here the method of obtaining the estimate of $\underline{x}$ is reduced to

estimating $\underline{\xi}(t)$ which reduces the order of the filter from $(n+p)$ to n since $\underline{z}_2$ is known data. Thus

$$\hat{\underline{x}}(t \mid t) = J_2(t)\underline{z}_2(t) + J_3(t)\hat{\underline{\xi}}(t \mid t) \qquad (11.16)$$

Referring to Equation 11.10 note that

$$\underline{\xi}(t) = H_3(t)\ \underline{x}(t) \qquad (11.17)$$

Differentiating this equation and substituting in Equations 11.8 and 11.15 using simplified notation yields

$$\dot{\underline{\xi}} = \dot{H}_3\underline{x} + H_3\dot{\underline{x}}$$

$$= \dot{H}_3(J_2\underline{z}_2 + J_3\underline{\xi}) + H_3[F(J_2\underline{z}_2 + J_3\underline{\xi}) + \underline{w}]$$

$$= [(\dot{H}_3 + H_3F)J_3]\ \underline{\xi} + (\dot{H}_3 + H_3F)J_2\underline{z}_2 + H_3\underline{w} \qquad (11.18)$$

This equation is equivalent to the general form of the plant model with a deterministic forcing function equal to $(\dot{H}_3 + H_3F)\ J_2\underline{z}_2$.

The revised plant in terms of $\underline{\xi}$ necessitates a new expression for measurement data $\underline{z}_1(t)$ as

$$\underline{z}_1 = H_1\underline{x} + \underline{v}$$

$$= H_1(J_2\underline{z}_2 + J_3\underline{\xi}) + \underline{v}$$

$$= H_1J_2\underline{z}_2 + H_1J_3\underline{\xi} + \underline{v} \qquad (11.19)$$

Define a new measurement of $\underline{z}_1$ modified by a known quantity $H_1J_2\underline{z}_2$ as

$$\underline{z} \overset{\Delta}{=} \underline{z}_1 - H_1J_2\underline{z}_2 \qquad (11.20)$$

then

$$\underline{z} = [H_1J_3]\ \underline{\xi} + \underline{v} \qquad (11.21)$$

The system formed by Equations 11.18, 11.20, and 11.21 form a new system which can be solved for the estimate of $\underline{\xi}(t)$ by previously derived Kalman-Bucy filter equations in Appendix A. After the estimate of $\underline{\xi}$ is determined the estimate of the augmented state $\underline{x}$ is obtained from Equation 11.16.

It is sometimes possible and usually more desirable to estimate the unaugmented state vector $\underline{x}(t)$ directly without having to use an intermediate step such as Equation 11.16. Returning to this problem let $H_3(t)$ in the transformation $T(t)$, Equation 11.11 be defined as

$$H_3 \overset{\Delta}{=} [I \quad 0] \tag{11.22}$$

and from Equation 11.7 let

$$H_2(t) = [M(t) \quad I] \tag{11.23}$$

then

$$T(t) \overset{\Delta}{=} \begin{bmatrix} H_2(t) \\ H_3 \end{bmatrix} = \begin{bmatrix} M(t) & I \\ I & 0 \end{bmatrix} \tag{11.24}$$

$$T^{-1}(t) \overset{\Delta}{=} [J_2 J_3(t)] = \begin{bmatrix} 0 & I \\ I & -M(t) \end{bmatrix} \tag{11.25}$$

Using this transformation with the augmented state vector in Equations 11.6 and 11.7 gives

$$\begin{bmatrix} \underline{z}(t) \\ \underline{\xi}(t) \end{bmatrix} = T(t) \begin{bmatrix} \underline{x}(t) \\ \underline{n}(t) \end{bmatrix}$$

$$= \begin{bmatrix} M(t) & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{n}(t) \end{bmatrix} \tag{11.26}$$

and the filtered state is

$$\underline{\xi}(t) \equiv \underline{x}(t) \qquad (11.27)$$

or in other words, the unaugmented $\underline{x}(t)$ is being estimated
directly. Stated another way, the set of Equations 11.18,
11.20 and 11.21 can now be used to estimate the unaugmented
state vector $\underline{x}(t)$ directly from the continuous Kalman-Bucy
equations derived in Appendix A where the transformation
matrix quantities are determined from the given system as
described by the augmented Equations 11.6 and 11.7.

It is necessary to prove that Equation 11.18 does in fact
describe $\underline{x}(t)$ in accordance with Equation 11.1. Note from
Equation 11.22 that

$$\dot{H}_3 = 0 \qquad (11.28)$$

Then using Equations 11.22, 11.25, 11.27, and 11.28 in 11.18
leads to

$$\dot{\underline{\xi}} = (0 + [F \ 0]) \begin{bmatrix} I \\ -M \end{bmatrix} \underline{\xi} + (0) \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{z}_2 + [F \ 0] \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{z}_2 + [I \ 0] \begin{bmatrix} \underline{w} \\ \underline{v} \end{bmatrix}$$

$$\dot{\underline{\xi}} = \qquad F\underline{\xi} \qquad + \qquad 0 \qquad + \qquad 0 \qquad + \quad \underline{w}$$

or

$$\dot{\underline{x}} = [F]\underline{x} + [\underline{w}] \qquad (11.29)$$

which is identical to Equation 11.1. The measurement as
given by Equation 11.20 is determined by differentiating
Equation 11.2 thus

$$\dot{\underline{z}} = M\dot{\underline{x}} + \dot{M}\underline{x} + \dot{\underline{n}}$$

$$= MF\underline{x} + M\underline{w} + \dot{M}\underline{x} + A\underline{n} + \underline{v} \qquad (11.30)$$

Subtracting $A\underline{z}$ from both sides after noting that $A\underline{z} = AM\underline{x} + A\underline{n}$ gives

$$\dot{\underline{z}} - A\underline{z} = MF\underline{x} + M\dot{\underline{x}} + A\underline{n} + \underline{v} + M\underline{w} - AM\underline{x} - A\underline{n}$$

$$= [MF - AM + \dot{M}]\ \underline{x} + M\underline{w} + \underline{v} \qquad (11.31)$$

define

$$Z(t) \overset{\Delta}{=} \dot{\underline{z}} - A(t)\underline{z} = [J]\underline{x} + [M\underline{w} + \underline{v}] \qquad (11.32)$$

$$J(t) \overset{\Delta}{=} MF - A(t)M + \dot{M} \qquad (11.33)$$

which is equivalent to Equation 11.21. Note that the new system defined by Equations 11.29 and 11.32 can now be solved by previously derived equations and that here plant noise $[\underline{w}]$ is correlated to measurement noise $[M\underline{w} + \underline{v}]$ which causes no difficulty.

Finally the estimate for $\underline{x}$ is given by

$$\dot{\hat{\underline{x}}} = F(t)\hat{\underline{x}} + K(t)[\underline{Z}(t) - \hat{\underline{Z}}(t)]$$

$$= F(t)\hat{\underline{x}} + K(t)[\dot{\underline{z}} - A(t)\underline{z} - J(t)\hat{\underline{x}}] \qquad (11.34)$$

For the new system defined by Equations 11.29 and 11.32 observe that the new plant noise covariance is

$$E\{\underline{w}(t)\ \underline{w}^T(t-\tau)\} = Q(t)\delta(\tau) \qquad (11.35)$$

The measurement noise covariance is

$$E\{[M\ \underline{w}(t) + \underline{v}(t)][M\ \underline{w}(t-\tau) + \underline{v}(t-\tau)]^T\} = M\ E\{\underline{w}(t)\underline{w}^T(t-\tau)\}M^T$$

$$+ E\{\underline{v}(t)\underline{w}^T(t-\tau)\}M^T + M\ E\{\underline{w}(t)\ \underline{v}^T(t-\tau)\} + E\{\underline{v}(t)\underline{v}^T(t-\tau)\}$$

$$= M\ Q(t)\ \delta(\tau)\ M^T + 0 + 0 + R(t)\ \delta(\tau) \qquad (11.36)$$

The new cross correlation between the new plant and new
measurement noise is

$$E\{[\underline{w}(t)][M\underline{w}(t-\tau)+\underline{v}(t-\tau)]^T\}=E\{\underline{w}(t)\underline{w}^T(t-\tau)\}M^T+E\{\underline{w}(t)\underline{v}^T(t-\tau)\}$$

$$= Q(t)\delta(\tau)M^T+0 \qquad\qquad (11.37)$$

Thus according to the old system given by Equations 10.1 and
10.2 and here denoted by primed quantities when noise cross
correlation exists it is given by

$$E\{\underline{w}'(t)\ \underline{v}'(t-\tau)\} = C'(t)\delta(\tau) \qquad\qquad (11.38)$$

and from Equations 11.35 and 11.38 in terms of primed correla-
tion matrices

$$Q'(t) = Q(t) \qquad\qquad (11.39)$$

$$R'(t) = M(t)\ Q(t)\ M(t)^T + R(t) \qquad\qquad (11.40)$$

$$C'(t) = Q(t)\ M^T(t) \qquad\qquad (11.41)$$

$$M'(t) = J(t) \qquad\qquad (11.42)$$

$$F'(t) = F(t) \qquad\qquad (11.43)$$

The optimal gain matrix K(t) which is known for the noise
cross-correlation case of the primed or old system can be
modified to represent the new system as

$$K(t) = [P(t)M'^T(t) + C'(t)]R'(t)^{-1}$$

$$= [P(t)J^T(t) + Q(t)M^T(t)][M(t)Q(t)M^T(t) + R(t)]^{-1}$$
$$\qquad\qquad (11.44)$$

Likewise this can be done to determine the error-covariance
matrix for the optimal gain case as given by the solution to
the matrix differential equation

$$\dot{P}(t) = F'(t) \ P(t) + P(t) \ F'^T(t) - [P(t) \ M'^T(t) + C'(t)]$$

$$R'^{-1}(t)[C'^T(t) + M'(t)P(t)] + Q'(t)$$

$$= F(t)P(t) + P(t)F^T(t) + Q(t)$$

$$- [P(t)J^T(t)+Q(t)M^T(t)][M(t)Q(t)M^T(t)+R(t)]^{-1}.$$

$$[M(t)Q^T(t)+J(t)P(t)]$$

$$= F(t) \ P(t) + P(t) \ F^T(t) + Q(t) - K(t) \ [M(t) \ Q(t) \ M^T(t)$$

$$+ R(t)] \ K^T(t) \qquad\qquad (11.45)$$

This concludes the generalized solution of the continuous Kalman-Bucy filter for the special case of colored measurement noise.